UNIVERSIDAD DE BUENOS AIRES
Facultad de Ciencias Exactas y Naturales
Departamento de Matemática

K-teoría algebraica y topológica de anillos de grupo

Tesis presentada para optar al título de Doctor de la
Universidad de Buenos Aires en el área Ciencias Matemáticas

Gisela Tartaglia

Director de tesis: Dr. Guillermo Cortiñas
Consejero de estudios: Dr. Guillermo Cortiñas

Buenos Aires, 2015

K-TEORÍA ALGEBRAICA Y TOPOLÓGICA DE ANILLOS DE GRUPO

Las conjeturas de isomorfismo prevén una descripción de la K-teoría (en sus diversas variantes) del producto cruzado $R \rtimes G$ de un grupo G con coeficientes en un anillo R equipado con una acción de G, en términos de topología algebraica. En esta tesis estudiamos diferentes versiones de las conjeturas con ideales de operadores como anillos de coeficientes.

Consideramos primero el morfismo de ensamble para la K-teoría algebraica con coeficientes en el anillo $\mathcal{S}$ de operadores de Schatten. Guoliang Yu probó que este morfismo es racionalmente inyectivo. Su prueba involucra la construcción de un cierto caracter de Chern que funciona para el caso particular con coeficientes $K(\mathcal{S})$. Aquí damos una demostración alternativa del resultado de Yu, formulándolo en términos de K-teoría homotópica y utilizando el caracter de Chern usual con valores en la homología cíclica.

Mostramos además que si G satisface la conjetura de isomorfismo racional para K-teoría homotópica con coeficientes en el álgebra de operadores de traza en un espacio de Hilbert, entonces también satisface la conjetura de Novikov para K-teoría algebraica y la parte de inyectividad racional de la conjetura de Farrell-Jones con coeficientes en cualquier cuerpo de números.

Finalmente probamos la validez de la conjetura de Farrell-Jones para un grupo a-T-menable G con coeficientes en un anillo de la forma $I \otimes (\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K})$, donde I es un G-anillo K-escisivo, $\mathfrak{A}$ es una G-C^*-álgebra y $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ es ideal de operadores compactos. Utilizando esto y el resultado de Higson y Kasparov sobre la validez de la conjetura de Baum-Connes con coeficientes para tales grupos, mostramos que si $\mathfrak{A}$ es estable y separable la K-teoría del producto cruzado algebraico $\mathfrak{A} \rtimes G$ coincide con la K-teoría de la C^*-álgebra plena $C^*(\mathfrak{A}, G)$.

Palabras clave: K-teoría, conjeturas de isomorfismo, ideales de operadores, homología, álgebras de grupo.

ALGEBRAIC AND TOPOLOGICAL K-THEORY FOR GROUP RINGS

The isomorphism conjectures predict a description of the K-theory (in its different variants) of the crossed product $R \rtimes G$ of a group G with coefficients in a ring R equipped with an action of G, in terms of algebraic topology. In this thesis we study different versions of the conjectures with operator ideals as coefficient rings.

We consider first the assembly map for algebraic K-theory with coefficients in the ring of Schatten operators $\mathcal{S}$. Guoliang Yu proved that this morphism is rationally injective. His proof involves the construction of a certain Chern character tailored to work with coefficients $K(\mathcal{S})$. Here we give a different proof of Yu's result; we formulate it in terms of homotopy K-theory and we use the usual Chern character with values in the cyclic homology.

We also show that if G satisfies the rational isomorphism conjecture for homotopy K-theory with coefficients in the algebra of trace-class operators in a Hilbert space, then it also satisfies the Novikov conjecture for algebraic K-theory and the rational injectivity part of the Farrell-Jones conjecture with coefficients in any number field.

Finally we prove the validity of the Farrell-Jones conjecture for an a-T-menable group G with coefficients in a ring of the form $I \otimes (\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K})$, where I is a K-excisive G-ring, $\mathfrak{A}$ is a G-C^*-algebra and $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N})$ is the ideal of compact operators. We use this and the result of Higson and Kasparov that the Baum-Connes conjecture with coefficients holds for such groups, to show that if $\mathfrak{A}$ is stable and separable, then the algebraic and the C^*-crossed products of G with $\mathfrak{A}$ have the same K-theory.

Keywords: K-theory, isomorphism conjectures, operator ideals, homology, group algebras.

AGRADECIMIENTOS

A Willie por todo lo que me enseñó en estos años, por su constante apoyo y paciencia.

Al CONICET por haberme dado la posibilidad de realizar este doctorado y al Departamento de Matemática de la UBA por brindarme un lugar de trabajo.

A Walter por bancarme siempre, por cuidarme y confiar en mí. A Juani por su alegría y su entusiasmo que me dan fuerzas para seguir adelante.

A Euge y Euge, por estar siempre, por hacerme reir tanto, por escucharme (zarpadas amigas encontré bo! ;)).

A mis viejos y a mis amigos por acompañarme.

A mis compañeros de oficina, a los profesores de la UBA, a mis compañeros de trabajo de la UNLP y a todos los que de alguna manera aportaron su granito de arena para que esta tesis se hiciera realidad.

Gracias!

A mis viejos.
A Walter y Juani.

CONTENTS

INTRODUCTION

Let G be a group; a *family* of subgroups of G is a nonempty family $\mathcal{F}$ closed under conjugation and under taking subgroups. A *G-space* is a simplicial set together with a G-action. If $\mathcal{F}$ is a family of subgroups of G and $f : X \to Y$ is an equivariant map of G-spaces, then we say that f is an *$\mathcal{F}$-equivalence* (resp. an *$\mathcal{F}$-fibration*) if the map between fixed point sets

$$f : X^H \to Y^H$$

is a weak equivalence (resp. a fibration) for every $H \in \mathcal{F}$. A G-space X is called a *$(G, \mathcal{F})$-complex* if the stabilizer of every simplex of X is in $\mathcal{F}$. The category of G-spaces can be equipped with a closed model structure where the weak equivalences (resp. the fibrations) are the $\mathcal{F}$-equivalences (resp. the $\mathcal{F}$-fibrations), (see [6, §1]). The $(G, \mathcal{F})$-complexes are the cofibrant objects in this model structure.

By a general construction of Davis and Lück (see [14]) any functor E from the category $\mathbb{Z}$-Cat of small $\mathbb{Z}$-linear categories to the category Spt of spectra which sends category equivalences to weak equivalences of spectra gives rise to an equivariant homology theory of G-spaces $X \mapsto H^G(X, E(R))$ for each unital G-ring R. If $H \subset G$ is a subgroup, then

$$H^G_*(G/H, E(R)) = E_*(R \rtimes H) \tag{1}$$

is just E_* evaluated at the crossed product ring. The *isomorphism conjecture* for the quadruple $(G, \mathcal{F}, E, R)$ asserts that if $\mathcal{E}(G, \mathcal{F}) \overset{\sim}{\twoheadrightarrow} pt$ is a $(G, \mathcal{F})$-cofibrant replacement of the point, then the induced map

$$H^G_*(\mathcal{E}(G, \mathcal{F}), E(R)) \to E_*(R \rtimes G) \tag{2}$$

–called *assembly map*– is an isomorphism. If G acts trivially on R we write $R[G]$ for $R \rtimes G$.

We are interested in computing the right hand side of the assembly map for different values of E. However, the left hand side $H^G_*(\mathcal{E}(G, \mathcal{F}), E(R))$ is much more accessible and the smaller $\mathcal{F}$ is, the easier it is to compute it using homological methods like spectral sequences, Mayer-Vietoris arguments and Chern characters. For the family $\mathcal{F} = \mathcal{A}ll$ of all subgroups, the assembly map (2) is always an isomorphism because the one point space is a model for $\mathcal{E}(G, \mathcal{A}ll)$. The goal is to have an isomorphism for a family which is as small as possible. The appropriate choice of $\mathcal{F}$ varies with E. For $E = K$, the nonconnective algebraic K-theory spectrum, one takes $\mathcal{F} = \mathcal{V}cyc$, the family of virtually cyclic subgroups; the isomorphism conjecture for $(G, \mathcal{V}cyc, K, R)$ is the K-theoretic *Farrell-Jones conjecture* with coefficients in R. If $E = KH$ is homotopy K-theory, one can equivalently take $\mathcal{F}$ to be either $\mathcal{V}cyc$ or the family $\mathcal{F}in$ of finite subgroups ([2, Thm. 2.4]). If E satisfies certain hypothesis, including excision, one can make sense of the map (2) when R is replaced by any, not necessarily unital ring A. These hypothesis are satisfied, for example, when $E = KH$. Under milder hypothesis, which are satisfied, for example, by $E = K$, (2) makes sense for those coefficient rings A which are *E-excisive*, i.e. those for which E satisfies excision (see [6] and Section 1.3 below).

Let $\mathfrak{A}$ be a G-C^*-algebra. The *Baum-Connes conjecture* is the analogue of the Farrell-Jones conjecture for the topological K-theory of reduced C^*-crossed products. It asserts that the assembly map

$$H^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathfrak{A})) \to K^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A}))$$

is a weak equivalence. Here $C^*_{\mathrm{red}}(G, \mathfrak{A})$ is the reduced C^*-algebra (see Appendix C) and $H^G(-, K^{\mathrm{top}}(\mathfrak{A}))$ is equivariant topological K-homology. The latter homology is characterized by

$$H^G_*(G/H, K^{\mathrm{top}}(\mathfrak{A})) = K^{\mathrm{top}}_*(C^*_{\mathrm{red}}(H, \mathfrak{A})).$$

This conjecture implies many other well-known and important conjectures. Examples are the Novikov Conjecture about the homotopy invariance of higher signatures and the Kadison Conjecture about idempotents in the reduced C^*-algebra of a torsion-free discrete group G.

Both the Farrell-Jones and the Baum-Connes conjectures have been proven for a large class of groups using a variety of different methods coming from operator theory, controlled topology and homotopy theory (see for example [3] for the case of Gromov hyperbolic groups and algebraic K-theory, and [20] for the case of a-T-menable groups and topological K-theory). Methods which have been developed for this purpose turned out to be fruitful in other contexts. It is worth mentioning that there are no counterexamples known to the Farrell-Jones conjecture and to the Baum-Connes conjecture with coefficients in $\mathfrak{A} = \mathbb{C}$.

In this thesis we study different versions of the isomorphism conjectures with operator ideals as coefficient rings. In Chapter 1 we recall some basic definitions, notations and results on equivariant homology theories of simplicial sets.

Let $\mathcal{B}$ be the ring of bounded operators in a complex, separable Hilbert space. In Chapter 2 we consider the following Farrell-Jones assembly map:

$$H^G_*(\mathcal{E}(G, \mathcal{V}cyc), K(\mathcal{S})) \to K_*(\mathcal{S}[G]). \tag{3}$$

Here $\mathcal{S} = \bigcup_{p>0} \mathcal{L}^p$, and $\mathcal{L}^p \lhd \mathcal{B}$ is the Schatten ideal of those compact operators whose sequence of singular values is p-summable. Guoliang Yu proved that the assembly map (3) is rationally injective ([36]). His proof involves the construction of a certain Chern character tailored to work with coefficients $\mathcal{S}$ and the use of some results about algebraic K-theory of operator ideals and about controlled topology and coarse geometry. In this chapter we give a different proof of Yu's result. Our proof uses the usual Chern character to cyclic homology. Like Yu's, it relies on results on algebraic K-theory of operator ideals, but no controlled topology or coarse geometry techniques are used. We formulate the result in terms of homotopy K-theory. In Theorem 2.3.1 we prove that the rational assembly map

$$H^G_*(\mathcal{E}(G, \mathcal{F}in), KH(\mathcal{L}^p)) \otimes \mathbb{Q} \to KH_*(\mathcal{L}^p[G]) \otimes \mathbb{Q} \tag{4}$$

is injective. Then we show that the latter map is equivalent to the assembly map considered by Yu, and thus obtain his result as Corollary 2.3.2. The results of this chapter appeared in our article [7].

In Chapter 3 we prove that if (4) is surjective for $p = 1$ and F is a number field, then the following assembly maps are injective:

$$H_*^G(\mathcal{E}(G,\{1\}),K(\mathbb{Z}))\otimes\mathbb{Q}\to K_*(\mathbb{Z}[G])\otimes\mathbb{Q} \tag{5}$$
$$H_*^G(\mathcal{E}(G,\mathcal{F}in),K(F))\otimes\mathbb{Q}\to K_*(F[G])\otimes\mathbb{Q}. \tag{6}$$

We remark that the K-theory Novikov conjecture asserts that the assembly (5) is injective for all G. Hence the validity of the rational isomorphism conjecture for KH with coefficients in $\mathcal{L}^1$ implies the validity of the Novikov conjecture for K-theory. The idea of the proof is to use an algebraic, equivariant version of Karoubi's multiplicative K-theory: given a unital $\mathbb{C}$-algebra A, we define groups $\kappa_n(A)$ $(n\in\mathbb{Z})$ which fit into a long exact sequence

$$HC_{n-1}(A/\mathbb{C})\to\kappa_n(A)\to KH_n(\mathcal{L}^1\otimes_{\mathbb{C}}A)\overset{\mathrm{Tr}Sch_n}{\to} HC_{n-2}(A/\mathbb{C}).$$

Here $HC(/\mathbb{C})$ is algebraic cyclic homology of $\mathbb{C}$-algebras, ch is the algebraic Connes-Karoubi Chern character with values in periodic cyclic homology HP, $S:HP\to HC[-2]$ is the periodicity operator and Tr is induced by the operator trace. We also introduce a character

$$\tau_n:K_n(A)\to\kappa_n(A)\quad(n\in\mathbb{Z}).$$

Both κ and τ have equivariant versions, so we can consider the rational assembly map for the family $\mathcal{F}cyc$ of finite cyclic subgroups

$$H_*^G(\mathcal{E}(G,\mathcal{F}cyc),\kappa(\mathbb{C}))\otimes\mathbb{Q}\to\kappa_*(\mathbb{C}[G])\otimes\mathbb{Q}. \tag{7}$$

We show that if (4) is surjective for $p=1$ then (7) is injective, and this implies the injectivity of (5) and (6). The results of this chapter appeared in our article [8].

Finally, in Chapter 4 we study the techniques used by Higson, Kasparov and Trout in [19], [20] and [21] to prove the Baum-Connes conjecture for a-T-menable groups, and we apply them to the algebraic case. More precisely, we prove the validity of the Farrell-Jones conjecture for an *a-T-menable* group G (see [15]) with coefficients in a ring of the form $I\otimes(\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K})$, where I is a K-excisive G-ring, $\mathfrak{A}$ is a G-C^*-algebra (or more generally, a G-bornolocal C^*-algebra as defined in Section 4.1), $\mathcal{K}=\mathcal{K}(\ell^2(\mathbb{N}))$ is the ideal of compact operators and $\underset{\sim}{\otimes}$ is the tensor product of C^*-algebras. Moreover, if we consider the following commutative diagram

$$\begin{tikzcd}
H_*^G(\mathcal{E}(G,\mathcal{F}in),K(\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K})) \arrow[r] \arrow[d] & K_*((\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K})\rtimes G) \arrow[d, "\alpha"] \\
H_*^G(\mathcal{E}(G,\mathcal{F}in),K^{\mathrm{top}}(\mathfrak{A})) \arrow[r] & K_*^{\mathrm{top}}(C^*_{\mathrm{red}}(G,\mathfrak{A}))
\end{tikzcd}$$

and we assume that $\mathfrak{A}$ is separable, we obtain as a corollary that α is an isomorphism. The results of this chapter appeared in our article [9].

Some general notation: If X is a locally compact Hausdorff space, we denote by $C_b(X)=C_b(X,\mathbb{C})$ the space of bounded continuous $\mathbb{C}$-valued functions on X, and by $C_c(X)$ and $C_0(X)$ respectively those functions in $C_b(X)$ which have compact supports and those which vanish at infinity. If R is a ring, we denote by $\otimes_R$ the *algebraic tensor product* over R. In case $R=\mathbb{Z}$, we omit it from the notation.

INTRODUCCIÓN

Sea G un grupo; una *familia* de subgrupos de G es una familia no vacía $\mathcal{F}$ cerrada por conjugación y por subgrupos. Un *G-espacio* es un conjunto simplicial equipado con una acción de G. Si $\mathcal{F}$ es una familia de subgrupos de G y $f : X \to Y$ es un morfismo equivariante de G-espacios, decimos que f es una *$\mathcal{F}$-equivalencia* (resp. una *$\mathcal{F}$-fibración*) si el morfismo entre los conjuntos de puntos fijos

$$f : X^H \to Y^H$$

es una equivalencia débil (resp. una fibración) para todo $H \in \mathcal{F}$. Un G-espacio X se llama *$(G, \mathcal{F})$-complejo* si el estabilizador de cada símplex de X está en $\mathcal{F}$. La categoría de G-espacios se puede dotar de una estructura de modelos cerrada donde las equivalencias débiles (resp. las fibraciones) son las $\mathcal{F}$-equivalencias (resp. las $\mathcal{F}$-fibraciones), (ver [6, §1]). Los $(G, \mathcal{F})$-complejos son los objetos cofibrantes en esta estructura de modelos.

Por una construcción general de Davis y Lück (ver [14]), todo funtor E de la categoría $\mathbb{Z}$-Cat de categorías $\mathbb{Z}$-lineales pequeñas en la categoría Spt de espectros que manda equivalencias de categorías en equivalencias débiles de espectros define una teoría de homología equivariante de G-espacios $X \mapsto H^G(X, E(R))$ para cada G-anillo unitario R. Para $H \subset G$ un subgrupo se obtiene

$$H^G_*(G/H, E(R)) = E_*(R \rtimes H), \tag{1'}$$

el valor de E_* en el producto cruzado algebraico. La *conjetura de isomorfismo* para la cuádrupla $(G, \mathcal{F}, E, R)$ afirma que si $\mathcal{E}(G, \mathcal{F}) \overset{\sim}{\twoheadrightarrow} pt$ es un $(G, \mathcal{F})$-reemplazo cofibrante del punto, entonces el morfismo inducido

$$H^G_*(\mathcal{E}(G, \mathcal{F}), E(R)) \to E_*(R \rtimes G) \tag{2'}$$

–llamado *morfismo de ensamble*– es un isomorfismo. Si G actúa trivialmente en R escribimos $R[G]$ en lugar de $R \rtimes G$.

Estamos interesados en calcular el lado derecho del morfismo de ensamble para diferentes valores de E. Sin embargo, el lado izquierdo $H^G_*(\mathcal{E}(G, \mathcal{F}), E(R))$ es mucho más accesible y a medida que $\mathcal{F}$ es más pequeña, es más fácil calcularlo usando métodos homológicos como sucesiones espectrales, Mayer-Vietoris y caracteres de Chern. Para la familia $\mathcal{F} = \mathcal{A}ll$ de todos los subgrupos, el morfismo de ensamble (2′) es siempre un isomorfismo ya que el punto es un modelo para $\mathcal{E}(G, \mathcal{A}ll)$. El objetivo es obtener un isomorfismo para una familia lo más pequeña posible. La elección apropiada de $\mathcal{F}$ varía con E. Para $E = K$, el espectro no-conectivo de la K-teoría algebraica, se toma $\mathcal{F} = \mathcal{V}cyc$, la familia de subgrupos virtualmente cíclicos; la conjetura de isomorfismo para $(G, \mathcal{V}cyc, K, R)$ es la *conjetura de Farrell-Jones* para K-teoría con coeficientes en R. Si $E = KH$ es la K-teoría homotópica, se puede considerar equivalentemente la familia $\mathcal{V}cyc$ o la familia $\mathcal{F}in$ de subgrupos finitos ([2, Teo. 2.4]). Si E satisface ciertas hipótesis, incluyendo escisión, tiene sentido considerar el morfismo (2′) cuando R es reemplazado por un anillo, no necesariamente unital, A. Estas hipótesis se satisfacen, por ejemplo, cuando

$E = KH$. Bajo hipótesis más suaves, que se satisfacen por ejemplo para $E = K$, (2′) tiene sentido para anillos E-*escisivos*, i.e. aquellos anillos para los que E satisface escisión (ver [6] y Sección 1.3).

Sea $\mathfrak{A}$ una G-C^*-álgebra. La *conjetura de Baum-Connes* es el análogo de la conjetura de Farrell-Jones para la K-teoría topológica de productos cruzados reducidos de C^*-álgebras. Esta conjetura afirma que el morfismo de ensamble

$$H^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathfrak{A})) \to K^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A}))$$

es una equivalencia débil. Aquí $C^*_{\mathrm{red}}(G, \mathfrak{A})$ es la C^*-álgebra reducida (ver Apéndice C) y $H^G(-, K^{\mathrm{top}}(\mathfrak{A}))$ es la K-homología topológica equivariante. Esta homología está caracterizada por

$$H^G_*(G/H, K^{\mathrm{top}}(\mathfrak{A})) = K^{\mathrm{top}}_*(C^*_{\mathrm{red}}(H, \mathfrak{A})).$$

Esta conjetura implica otras conjeturas importantes. Por ejemplo, la Conjetura de Novikov sobre la invarianza homotópica de las signaturas de orden superior y la Conjetura de Kadison sobre idempotentes en la C^*-álgebra reducida de un grupo discreto G sin torsión.

Tanto la conjetura de Farrell-Jones como la de Baum-Connes fueron probadas para una gran clase de grupos usando diferentes métodos provenientes de teoría de operadores, topología controlada y teoría de homotopía (ver por ejemplo [3] para el caso de grupos hiperbólicos de Gromov y K-teoría algebraica, y [20] para el caso de grupos a-T-menables y K-teoría topológica). Los métodos que se desarrollaron para este propósito resultaron fructíferos en otros contextos. Cabe mencionar que no se conocen contraejemplos de la conjetura de Farrell-Jones ni de la conjetura de Baum-Connes con coeficientes en $\mathfrak{A} = \mathbb{C}$.

En esta tesis estudiamos diferentes versiones de las conjeturas de isomorfismo con ideales de operadores como anillos de coeficientes. En el Capítulo 1 recordamos algunas definiciones básicas, notaciones y resultados sobre teorías de homología equivariantes de conjuntos simpliciales.

Sea $\mathcal{B}$ el anillo de operadores acotados en un espacio de Hilbert complejo, separable, de dimensión infinita. En el Capítulo 2 consideramos el siguiente morfismo de ensamble de Farrell-Jones:

$$H^G_*(\mathcal{E}(G, \mathcal{V}cyc), K(\mathcal{S})) \to K_*(\mathcal{S}[G]). \tag{3′}$$

Aquí $\mathcal{S} = \bigcup_{p>0} \mathcal{L}^p$, y $\mathcal{L}^p \lhd \mathcal{B}$ es el ideal de Schatten formado por los operadores compactos cuya sucesión de valores singulares es p-sumable. Guoliang Yu probó que el morfismo de ensamble (3′) es racionalmente inyectivo ([36]). Su prueba involucra la construcción de un cierto caracter de Chern que funciona para el caso particular con coeficientes $K(\mathcal{S})$ y en resultados sobre K-teoría algebraica de ideales de operadores y sobre topología controlada y geometría de escala. En este capítulo damos una demostración alternativa del resultado de Yu. Nuestra prueba utiliza el caracter de Chern usual con valores en la homología cíclica. Al igual que Yu, nos basamos en resultados de K-teoría algebraica de ideales de operadores, pero no utilizamos herramientas de topología controlada ni de geometría de escala. Formulamos el resultado en términos de K-teoría homotópica. En el Teorema 2.3.1 probamos que el morfismo de ensamble racional

$$H^G_*(\mathcal{E}(G, \mathcal{F}in), KH(\mathcal{L}^p)) \otimes \mathbb{Q} \to KH_*(\mathcal{L}^p[G]) \otimes \mathbb{Q} \tag{4′}$$

es inyectivo. Luego mostramos que este morfismo es equivalente al morfismo de ensamble considerado por Yu, y obtenemos su resultado en el Corolario (2.3.2). Los resultados de este capítulo están publicados en nuestro artículo [7].

En el Capítulo 3 probamos que si (4′) es suryectiva para $p = 1$ y F es un cuerpo de números, los siguientes morfismos de ensamble resultan inyectivos:

$$H_*^G(\mathcal{E}(G, \{1\}), K(\mathbb{Z})) \otimes \mathbb{Q} \to K_*(\mathbb{Z}[G]) \otimes \mathbb{Q} \tag{5'}$$

$$H_*^G(\mathcal{E}(G, \mathcal{F}in), K(F)) \otimes \mathbb{Q} \to K_*(F[G]) \otimes \mathbb{Q}. \tag{6'}$$

Observemos que la Conjetura de Novikov para K-teoría afirma que la aplicación de ensamble (5′) es inyectiva para todo G. Por lo tanto, la validez de la conjetura de isomorfismo racional para KH con coeficientes en $\mathcal{L}^1$ implica la validez de la Conjetura de Novikov para K-teoría. La idea de la demostración es usar una versión algebraica y equivariante de la K-teoría multiplicativa de Karoubi: dada una $\mathbb{C}$-álgebra unital A, definimos grupos $\kappa_n(A)$ $(n \in \mathbb{Z})$ que forman parte de una sucesión exacta larga

$$HC_{n-1}(A/\mathbb{C}) \to \kappa_n(A) \to KH_n(\mathcal{L}^1 \otimes_{\mathbb{C}} A) \stackrel{\mathrm{Tr}Sch_n}{\to} HC_{n-2}(A/\mathbb{C}).$$

Aquí $HC(/\mathbb{C})$ denota la homología cíclica de $\mathbb{C}$-álgebras, ch es el caracter de Chern de Connes-Karoubi que toma valores en la homología cíclica periódica HP, $S : HP \to HC[-2]$ es el operador de periodicidad y Tr es la aplicación inducida por el operador de traza. Introducimos además un caracter

$$\tau_n : K_n(A) \to \kappa_n(A) \quad (n \in \mathbb{Z}).$$

Tanto κ como τ tienen versiones equivariantes; esto nos permite considerar la aplicación de ensamble racional para la familia $\mathcal{F}cyc$ de subgrupos cíclicos finitos

$$H_*^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q} \to \kappa_*(\mathbb{C}[G]) \otimes \mathbb{Q}. \tag{7'}$$

Mostramos que si (4′) es suryectiva para $p = 1$ entonces (7′) es inyectiva, y esto implica la inyectividad de (5′) y (6′). Los resultados de este capítulo están publicados en nuestro artículo [8].

Finalmente, en el Capítulo 4 estudiamos las técnicas utilizadas por Higson, Kasparov y Trout en [19], [20] y [21] para probar la conjetura de Baum-Connes para grupos a-T-menables, y las aplicamos al caso algebraico. Más precisamente, probamos la validez de la conjetura de Farrell-Jones para un grupo *a-T-menable* G (ver [15]) con coeficientes en un anillo de la forma $I \otimes (\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K})$, donde I es un G-anillo K-escisivo, $\mathfrak{A}$ es una G-C^*-álgebra (o más generalmente, una G-C^*-álgebra bornolocal como se define en la Sección 4.1), $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ es el ideal de operadores compactos y $\underset{\sim}{\otimes}$ es el producto tensorial de C^*-álgebras. Más aún, si consideramos el siguiente diagrama conmutativo

$$\begin{array}{ccc} H_*^G(\mathcal{E}(G, \mathcal{F}in), K(\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K})) & \longrightarrow & K_*((\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K}) \rtimes G) \\ \downarrow & & \downarrow \alpha \\ H_*^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathfrak{A})) & \longrightarrow & K_*^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A})) \end{array}$$

y asumimos que $\mathfrak{A}$ es separable, obtenemos como corolario que α es un isomorfismo. Los resultados de este capítulo están publicados en nuestro artículo [9].

Notación general: si X es un espacio localmente compacto Hausdorff, denotamos por $C_b(X) = C_b(X, \mathbb{C})$ al espacio de funciones continuas y acotadas de X en $\mathbb{C}$, y por $C_c(X)$ y $C_0(X)$ respectivamente aquellas funciones en $C_b(X)$ de soporte compacto y aquellas que tienden a cero en el infinito. Si R es un anillo, denotamos por $\otimes_R$ al *producto tensorial algebraico* sobre R. En el caso particular en que $R = \mathbb{Z}$, lo omitimos de la notación.

1. PRELIMINARIES

In this section we recall some basic definitions and state preliminary results that are useful for the rest of the thesis. We define the category Spt of simplicial spectra, and also the homotopy groups and weak equivalences of spectra. If k is a commutative unital ring, we write k-Cat for the category of small k-linear categories. Then we consider a group G and a functor E from k-Cat into Spt, which satisfies the *Standing Assumptions* 1.3.2, and recall the construction of an equivariant E-homology theory of G-spaces.

1.1 Simplicial spectra

Let S^1 be the simplicial circle $\Delta[1]/\partial\Delta[1]$, obtained by identifying the two vertices of $\Delta[1]$. A *spectrum* is a sequence $E = \{{}_nE : n \geq 1\}$ of pointed simplicial sets and structure maps $\sigma_n : S^1 \wedge {}_nE \to {}_{n+1}E$. A map of spectra $f : E \to E'$ is a sequence of pointed maps $f_n : {}_nE \to {}_nE'$ such that the diagram

$$\begin{tikzcd} S^1 \wedge {}_nE \arrow[r, "\sigma_n"] \arrow[d, "id \wedge f"'] & {}_{n+1}E \arrow[d, "f_{n+1}"] \\ S^1 \wedge {}_nE' \arrow[r, "\sigma'_{n+1}"'] & {}_{n+1}E' \end{tikzcd}$$

is commutative $\forall n \geq 1$.

A homotopy of maps of spectra $f_k : E \to E'$ is a map of spectra $H : \Delta[1]_+ \wedge E \to E'$ whose composition with the inclusion $i_k \wedge id : E \cong \Delta[0] \wedge E \to \Delta[1]_+ \wedge E$ is f_k for $k = 0, 1$.

We write Spt for the category of spectra and maps of spectra, and HoSpt for the corresponding homotopy category.

The homotopy groups of a spectrum are defined by

$$\pi_n(E) = \operatorname*{colim}_{k \to \infty} \pi_{n+k}({}_kE),$$

where the system is given by the composition

$$\pi_{n+k}({}_kE) \xrightarrow{S} \pi_{n+k+1}(S^1 \wedge {}_kE) \xrightarrow{\sigma_{k*}} \pi_{n+k+1}({}_{k+1}E),$$

of the suspension homomorphism and the homomorphism induced by the structure map. We will write E_n for $\pi_n(E)$. A *weak homotopy equivalence* of spectra is a map of spectra $f : E \to F$ inducing an isomorphism on all homotopy groups.

Let $\mathcal{C}$ be a category and $E, F : \mathcal{C} \to$ Spt functorial spectra. A (natural) *map* $f : E \xrightarrow{\sim} F$ is a zig-zag of natural maps

$$E = Z_0 \xrightarrow{f_1} Z_1 \xleftarrow{f_2} Z_2 \xrightarrow{f_3} \dots Z_n = F$$

such that each right to left arrow f_i is an object-wise weak equivalence. If also the left to right arrows are object-wise weak equivalences, then we say that f is a *weak equivalence* or simply an *equivalence*. If E and F are spectra, we write $E \oplus F$ for their wedge or coproduct, i.e. ${}_n(E \oplus F) = {}_nE \vee {}_nF$. The Dold-Kan correspondence associates a spectrum to every chain complex of abelian groups. Although our notation does not distinguish a chain complex from the spectrum associated to it, it will be clear from the context which of the two we are referring to.

1.2 *G-simplicial sets*

A G-simplicial set X is a simplicial set with a simplicial action of G, i.e. at each level, X_n is a G-set and the actions are compatible with the simplicial structure maps. Any G-simplicial set is the colimit of its cells

$$X = \operatorname*{colim}_{G/H \times \Delta^n \to X} G/H \times \Delta^n.$$

The n-skeleton of a G-simplicial set can be obtained by attaching equivariant cells to X_{n-1} as the following push out diagram shows

$$\begin{tikzcd}
\coprod_{i \in I_n} G/H_i \times \partial\Delta^n \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i \in I_n} G/H_i \times \Delta^n \arrow[r] & X_n
\end{tikzcd}$$

A *family* of subgroups of G is a nonempty family $\mathcal{F}$ closed under conjugation and under taking subgroups. A G-simplicial set is a $(G, \mathcal{F})$-*complex* if $H_i \in \mathcal{F}$, $\forall i \in I_n$, $\forall n \in \mathbb{N}$. We will also use the term G-*space* for a G-simplicial set.

1.3 *Equivariant homology of simplicial sets*

Let k be a commutative unital ring. A k-*linear category* is a small category enriched over the category of k-modules. We write k-Cat for the category of k-linear categories and k-linear functors. Observe that, by regarding a unital k-algebra as k-linear category with one object, we obtain a fully faithful embedding of the category of unital k-algebras into k-Cat. Let $E : k$-Cat $\to$ Spt be a functor. If R is a unital k-algebra and $I \lhd R$ is a k-ideal, we put

$$E(R : I) := \mathrm{hofiber}(E(R) \to E(R/I)).$$

Thus if we assume $E(0) \xrightarrow{\sim} *$, we have $E(R : R) \xrightarrow{\sim} E(R)$. We say that E is *finitely additive* if the canonical map

$$E(\mathcal{C}) \oplus E(\mathcal{D}) \to E(\mathcal{C} \oplus \mathcal{D})$$

is an equivalence. We assume from now on that E is finitely additive. If A is a not necessarily unital k-algebra, write $\tilde{A}_k$ for its unitization; this is the k-module $A \oplus k$ equipped with the following multiplication:

$$(a, m)(b, n) = (ab + mb + na, mn).$$

Put

$$E(A) := E(\tilde{A}_k : A).$$

If A happens to be unital, we have two definitions for $E(A)$; they are equivalent by [4, Lemma 1.1]. A not necessarily unital ring A is called *E-excisive* if for any embedding $A \lhd R$ as an ideal of a unital k-algebra, the canonical map

$$E(A) \to E(R : A)$$

is an equivalence.

Let $\mathcal{C} \in k$-Cat, consider the k-module

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{x,y \in \mathcal{C}} \hom_{\mathcal{C}}(x, y).$$

If $f \in \mathcal{A}(\mathcal{C})$ write $f_{a,b}$ for the component in $\hom_{\mathcal{C}}(b, a)$. The following multiplication law

$$(fg)_{a,b} = \sum_{c \in \mathrm{ob}\mathcal{C}} f_{a,c} g_{c,b}$$

makes $\mathcal{A}(\mathcal{C})$ into an associative k-algebra, which is unital if and only if $\mathrm{ob}\mathcal{C}$ is finite. Whatever the cardinal of $\mathrm{ob}\mathcal{C}$ is, $\mathcal{A}(\mathcal{C})$ is always a ring with *local units*, i.e. a filtering colimit of unital rings. We call $\mathcal{A}(\mathcal{C})$ the *arrow ring* of $\mathcal{C}$. If $F : \mathcal{C} \mapsto \mathcal{D}$ is a k-linear functor which is injective on objects, then it defines a homomorphism $\mathcal{A}(F) : \mathcal{A}(\mathcal{C}) \to \mathcal{A}(\mathcal{D})$ by the rule $\alpha \mapsto F(\alpha)$. Hence we may regard $\mathcal{A}$ as a functor

$$\mathcal{A} : \text{inj-}k\text{-Cat} \to k\text{-Alg} \tag{1.3.1}$$

from the category of k-linear categories and functors which are injective on objects, to the category of k-algebras. However $\mathcal{A}(F)$ is not defined for general k-linear F.

If A is any, not necessarily unital G-k-algebra, the *algebraic crossed product* $A \rtimes G$ is the tensor product $A \otimes \mathbb{Z}[G]$ equipped with the twisted product

$$(a \rtimes g)(b \rtimes f) = ag(b) \rtimes gf, \qquad \forall a, b \in A, \forall g, f \in G.$$

Standing Assumptions 1.3.2. We shall henceforth assume that $E : k\text{-Cat} \to \mathrm{Spt}$ satisfies each of the following.

i) Every algebra with local units is E-excisive.

ii) If H is a group and A an E-excisive H-algebra, then $A \rtimes H$ is E-excisive.

iii) Let X be a set, write M_X for the ring of all matrices $(z_{x,y})_{x,y \in X \times X}$ with integer coefficients, only finitely many of which are nonzero. If A is E-excisive and $x \in X$, then $M_X A = M_X \otimes A$ is E-excisive, and E sends the map $A \to M_X A$, $a \mapsto e_{x,x} a$ to a weak equivalence.

iv) There is a natural weak equivalence $E(\mathcal{A}(\mathcal{C})) \xrightarrow{\sim} E(\mathcal{C})$ of functors inj-k-Cat $\to$ Spt.

v) Let A and B be algebras, and let $C = A \oplus B$ be their direct sum, with coordinate-wise multiplication. Then C is E-excisive if and only if both A and B are. Moreover if these equivalent conditions are satisfied, then the map $E(A) \oplus E(B) \to E(C)$ is an equivalence.

Examples 1.3.3. The (nonconnective) K-theory spectrum K satisfies the standing assumptions for $k = \mathbb{Z}$ as well as for any field of characteristic zero ([6, Prop. 4.3.1, Prop. 6.4]). The homotopy K-theory spectrum KH is *excisive*, i.e. every ring is KH-excisive [34]. Furthermore it satisfies the assumptions over any ground ring k ([6, Prop. 5.5]). A k-linear category $\mathcal{C}$ has associated a canonical cyclic k-module $C(\mathcal{C}/k)$ ([28]) with

$$C(\mathcal{C}/k)_n = \bigoplus_{(c_0,\dots,c_n)\in \mathrm{ob}\mathcal{C}^{n+1}} \hom_{\mathcal{C}}(c_1, c_0) \otimes_k \cdots \otimes_k \hom_{\mathcal{C}}(c_0, c_n).$$

The Hochschild, cyclic, negative cyclic and periodic cyclic homology of $\mathcal{C}$ over k are the respective homologies of $C(\mathcal{C}/k)$; they are denoted $HH(/k)$, $HC(/k)$, $HN(/k)$ and $HP(/k)$. Both $HH(/k)$ and $HC(/k)$ satisfy the assumptions above when k is any field [6, Prop. 6.4]. If k is a field of characteristic zero, then $HP(/k)$ is excisive [13]; furthermore, it satisfies the standing assumptions because $HC(/k)$ does. It follows that also $HN(/k)$ satisfies the assumptions.

Crossed products 1.3.4. Let $\mathcal{G}$ be a groupoid, i.e. a small category in which every morphism is an isomorphism, and let R be a unital k-algebra. An *action* of $\mathcal{G}$ on R is a functor $\rho : \mathcal{G} \to k\text{-Alg}_1$ such that $\rho(x) = R$ for all $x \in ob\mathcal{G}$. If $\rho(g) = id_R$ for all arrows $g \in \mathcal{G}$, the action is called *trivial.* Whenever ρ is fixed, we omit it from our notation, and write

$$g(r) = \rho(g)(r)$$

for $g \in ar\mathcal{G}$ and $r \in R$. Given a triple $(\mathcal{G}, \rho, R)$ we consider the small k-linear category $R \rtimes \mathcal{G}$ whose objects are those of $\mathcal{G}$ and

$$hom_{R\rtimes\mathcal{G}}(x, y) = R \otimes \mathbb{Z}[hom_{\mathcal{G}}(x, y)].$$

If $r \in R$ and $f \in hom_{\mathcal{G}}(x, y)$, we write $r \rtimes f$ for $r \otimes f$. Composition is defined by the rule

$$(r \rtimes f).(s \rtimes g) = rf(s) \rtimes fg$$

where f and g are composable arrows in $\mathcal{G}$. In case the action of $\mathcal{G}$ on R is trivial, we also write $R[\mathcal{G}]$ for $R \rtimes \mathcal{G}$.

Let G be a group, and S a G-set. Write $\mathcal{G}^G(S)$ for its *transport groupoid.* By definition $\mathrm{ob}\mathcal{G}^G(S) = S$, and $\hom_{\mathcal{G}^G(S)}(s, t) = \{g \in G : g \cdot s = t\}$. We write Or$G$ for the orbit category of G; its objects are the G-sets G/H, $H \subset G$ a subgroup; its homomorphisms are the G-equivariant maps. If R is a unital G-k-algebra, the rule $G/H \mapsto R \rtimes \mathcal{G}^G(G/H)$ defines a functor Or$G \to k$-Cat.

Remark 1.3.5. We put

$$\mathcal{A}(A \rtimes \mathcal{G}^G(G/H)) = \ker(\mathcal{A}(\tilde{A}_k \rtimes \mathcal{G}^G(G/H)) \to \mathcal{A}(k \rtimes \mathcal{G}^G(G/H))).$$

Note that $\mathcal{A}(A \rtimes \mathcal{G}^G(G/H))$ is always defined, even though $A \rtimes \mathcal{G}^G(G/H)$ is not. Moreover, by [6, Lemma 3.2.6] there is an isomorphism natural in A

$$\mathcal{A}(A \rtimes \mathcal{G}^G(G/H)) \xrightarrow{\sim} M_{G/H}(A \rtimes H). \tag{1.3.6}$$

If R is a unital G-algebra and $A \triangleleft R$ is a k-ideal, closed under the action of G, put

$$E(R \rtimes \mathcal{G}^G(G/H) : A \rtimes \mathcal{G}^G(G/H)) = \operatorname{hofiber}(E(R \rtimes \mathcal{G}^G(G/H)) \to E((R/A) \rtimes \mathcal{G}^G(G/H))).$$

The *G-equivariant E-homology* of a G-space X with coefficients in $(R : A)$ is the coend

$$H^G(X, E(R : A)) = \int^{\mathrm{Or}G} X_+^H \wedge E(R \rtimes \mathcal{G}^G(G/H) : A \rtimes \mathcal{G}^G(G/H)).$$

The spectrum $H^G(X, E(R : A))$ is a simplicial set version of the Davis-Lück equivariant homology spectrum associated with E ([14],[6]). For a G-algebra A we write

$$E(A \rtimes \mathcal{G}^G(G/H)) = E(\tilde{A}_k \rtimes \mathcal{G}^G(G/H) : A \rtimes \mathcal{G}^G(G/H))$$
$$\text{and } H^G(X, E(A)) = H^G(X, E(\tilde{A}_k : A)).$$

If A is unital, the two definitions of $E(A \rtimes \mathcal{G}^G(G/H))$ and $H^G(X, E(A))$ are weakly equivalent, by [6, Proposition 3.3.9(a)]. If A is E-excisive and $A \triangleleft R$ is an ideal embedding into a unital G-algebra R, then by [6, Proposition 3.3.12], the canonical map of $\mathrm{Or}G$-spectra is a weak equivalence

$$E(A \rtimes \mathcal{G}^G(G/H)) \xrightarrow{\sim} E(R \rtimes \mathcal{G}^G(G/H) : A \rtimes \mathcal{G}^G(G/H)). \tag{1.3.7}$$

For any G-algebra A we have a weak equivalence, natural in A

$$E(\tilde{A}_k \rtimes H : A \rtimes H) \xrightarrow{\sim} E(A \rtimes \mathcal{G}^G(G/H)). \tag{1.3.8}$$

If A is E-excisive, we furthermore have

$$E(A \rtimes H) \xrightarrow{\sim} E(\tilde{A}_k \rtimes H : A \rtimes H). \tag{1.3.9}$$

By [6, Prop. 3.3.9], if

$$0 \to A' \to A \to A" \to 0$$

is an extension of E-excisive G-algebras, then

$$E(A' \rtimes \mathcal{G}^G(-)) \to E(A \rtimes \mathcal{G}^G(-)) \to E(A" \rtimes \mathcal{G}^G(-)) \tag{1.3.10}$$

and

$$H^G(X, E(A')) \to H^G(X, E(A)) \to H^G(X, E(A")) \tag{1.3.11}$$

are homotopy fibrations. More generally, we have the following proposition.

Proposition 1.3.12. *Let*

$$0 \to A' \to A \to A'' \to 0$$

be an exact sequence of G-algebras. Assume that A' is E-excisive. Let X be a G-space. Then

$$H^G(X, E(A')) \to H^G(X, E(A)) \to H^G(X, E(A''))$$

is a homotopy fibration sequence.

Proof. It suffices to prove the proposition for X of the form G/H where $H \subset G$ is a subgroup. We have a homotopy commutative diagram with homotopy fibration rows

$$\begin{tikzcd}
E(A \rtimes \mathcal{G}^G(G/H)) \arrow[r] \arrow[d] & E(\tilde{A}_k \rtimes \mathcal{G}^G(G/H)) \arrow[r] \arrow[d] & E(k[\mathcal{G}^G(G/H)]) \arrow[d, equal] \\
E(A'' \rtimes \mathcal{G}^G(G/H)) \arrow[r] & E(\tilde{A}''_k \rtimes \mathcal{G}^G(G/H)) \arrow[r] & E(k[\mathcal{G}^G(G/H)])
\end{tikzcd}$$

It follows that the homotopy fiber of the first vertical map is weakly equivalent to that of the middle map, which in turn is weakly equivalent to $E(A' \rtimes \mathcal{G}^G(G/H))$, by (1.3.7). $\square$

1.4 Resumen

En esta sección recordamos definiciones básicas y presentamos algunos resultados preliminares que usaremos a lo largo de la tesis. Definimos la categoría Spt de espectros simpliciales, y también los grupos de homotopía y las equivalencias débiles de espectros. Dado un anillo unital k, llamamos k-Cat a la categoría cuyos objetos son las categorías pequeñas k-lineales. Luego consideramos un grupo G y un funtor E de k-Cat en Spt, que satisface las *Standing Assumptions* 1.3.2 y recordamos la definición de una teoría de E-homología equivariante de G-espacios.

2. SCHATTEN IDEALS AND ASSEMBLY MAPS IN K-THEORY

Let $\mathcal{B}$ be the ring of bounded operators in an infinite dimensional, separable, complex Hilbert space. Recall that the *singular values* of a compact operator $T \in \mathcal{B}$ are the square roots of the eingenvalues of the positive self-adjoint operator T^*T. Let $p > 0$, we write $\mathcal{L}^p \lhd \mathcal{B}$ for the Schatten ideal of those compact operators whose sequence of singular values is p-summable. Let $\mathcal{S} = \bigcup_{p>0} \mathcal{L}^p$ be the ring of all Schatten operators. By [10, Thm. 8.2.1] (see also [35, Thm. 4]), $\mathcal{S}$ is K-excisive. Thus the assembly map (2) with coefficients $K(\mathcal{S})$ makes sense; Guoliang Yu proved in [36] that it is rationally injective. His proof involves the construction of a certain Chern character tailored to work with coefficients $\mathcal{S}$ and the use of some results about algebraic K-theory of operator ideals ([10], [35]), and about controlled topology and coarse geometry from [1] and [29].

In this chapter we give a different proof of Yu's result. Our proof uses the usual Chern character to cyclic homology. Like Yu's, it relies on results about algebraic K-theory of operator ideals from [10] and [35], but no controlled topology or coarse geometry techniques are used. We formulate the result in terms of KH; we prove:

Theorem 2.1. *Let $p > 0$ and G a group. Then the rational assembly map*

$$H_*^G(\mathcal{E}(G, \mathcal{F}in), KH(\mathcal{L}^p)) \otimes \mathbb{Q} \to KH_*(\mathcal{L}^p[G]) \otimes \mathbb{Q}$$

is injective.

Yu's result follows as a corollary.

Corollary 2.2. ([36, Thm. 1.1]). *Let G be any group and let $\mathcal{S} = \bigcup_{p>0} \mathcal{L}^p$ be the ring of all Schatten operators. Then the rational assembly map*

$$H_*^G(\mathcal{E}(G, \mathcal{V}cyc), K(\mathcal{S})) \otimes \mathbb{Q} \to K_*(\mathcal{S}[G]) \otimes \mathbb{Q}$$

is injective.

The proof of the corollary makes it clear that the two assembly maps are isomorphic, so it really is the same result.

The rest of this chapter is organized as follows. In Section 2.1 we show that if X is a $(G, \mathcal{F}in)$-complex and $k \supset \mathbb{Q}$ a field, then

$$H_n^G(X, HP(k/k)) = \bigoplus_{p \in \mathbb{Z}} H_{n+2p}^G(X, HH(k/k)).$$

We use this to show in Proposition 2.1.5 that the assembly map

$$H_n^G(\mathcal{E}(G,\mathcal{F}in), HP(k/k)) \to HP_n(k[G]/k) \tag{2.3}$$

is injective for every group G. In Section 2.2 we consider the Connes-Karoubi Chern character

$$ch : H^G(X, KH(A)) \to H^G(X, HP(A/k))$$

defined in [6, §8]. We show in Proposition 2.2.5 that the composite of ch with the operator trace gives an equivalence

$$c : H^G(X, KH(\mathcal{L}^1)) \otimes \mathbb{C} \xrightarrow{\sim} H^G(X, HP(\mathbb{C}/\mathbb{C}))$$

for every $(G,\mathcal{F}in)$-complex X. From this and the fact that $KH_*(\mathcal{L}^1) \cong KH_*(\mathcal{L}^p)$ we deduce –in Corollary 2.2.7– that a similar equivalence which we also call c holds for every $p > 0$:

$$c : H^G(X, KH(\mathcal{L}^p)) \otimes \mathbb{C} \xrightarrow{\sim} H^G(X, HP(\mathbb{C}/\mathbb{C})). \tag{2.4}$$

Section 2.3 is concerned with Theorem 2.1, which we prove in Theorem 2.3.1. The proof uses (2.4) and the injectivity of (2.3). Yu's result 2.2 is proved in Corollary 2.3.2.

These results were published in [7].

2.1 Equivariant periodic cyclic homology of $(G,\mathcal{F}in)$-complexes

Proposition 2.1.1. *Let X be a $(G,\mathcal{F}in)$-complex, and let $k \supset \mathbb{Q}$ be a field. There is a natural weak equivalence*

$$\bigoplus_{p\in\mathbb{Z}} H^G(X, HH(k/k))[2p] \xrightarrow{\sim} H^G(X, HP(k/k)).$$

Proof. It suffices to show that there is a weak equivalence

$$\bigoplus_{p\in\mathbb{Z}} H^G(G/H, HH(k/k))[2p] \xrightarrow{\sim} H^G(G/H, HP(k/k))$$

for $H \in \mathcal{F}in$, natural with respect to G-equivariant maps. In particular we may restrict to proving the proposition for X a discrete, G-finite $(G,\mathcal{F}in)$-complex. The cyclic module $C(k[\mathcal{G}^G(X)]/k)$ decomposes into a direct sum of cyclic modules [6, 7.1]

$$C(k[\mathcal{G}^G(X)]/k) = \bigoplus_{(g)\in \mathrm{con}G} C^{(g)}(k[\mathcal{G}^G(X)]/k).$$

Here the direct sum runs over the set con$G$ of conjugacy classes of elements of G. Because X is a $(G,\mathcal{F}in)$-complex, $C^{(g)}(k[\mathcal{G}^G(X)]/k) = 0$ for g of infinite order. So assume g is of finite order. Let $Z_g \subset G$ be the centralizer subgroup. By [6, Lemma 7.2] (see also [27, Cor. 9.12]), there is a natural quasi-isomorphism of cyclic k-modules

$$H(Z_g, k[X^g]) \to C^{(g)}(k[\mathcal{G}^G(X)]/k).$$

Here the domain has the cyclic structure given by

$$t_n : H(Z_g, k[X^g])_n = k[Z_g]^{\otimes n} \otimes_k k[X^g] \to H(Z_g, k[X^g])_n$$
$$t_n(z_1 \otimes \cdots \otimes z_n \otimes x) = (z_1 \cdots z_n)^{-1} g \otimes z_1 \otimes \cdots \otimes z_{n-1} \otimes z_n(x).$$

Let $\langle g \rangle \subset Z_g$ be the cyclic subgroup. A Serre spectral sequence argument (see the proof of [33, Lemma 9.7.6]) shows that the projection

$$H(Z_g, k[X^g]) \to H(Z_g/\langle g \rangle, k[X^g])$$

is a quasi-isomorphism of cyclic k-modules. Summing up we have a natural zig-zag of quasi-isomorphisms of cyclic modules

$$\bigoplus_{(g)\in \mathrm{con}_f G} H(Z_g/\langle g \rangle, k[X^g]) \xleftarrow{\sim} \bigoplus_{(g)\in \mathrm{con}_f G} H(Z_g, k[X^g]) \xrightarrow{\sim} C(k[\mathcal{G}^G(X)]/k). \tag{2.1.2}$$

Here $\mathrm{con}_f G$ is the set of conjugacy classes of elements of finite order. We remark that, because X is a finite $(G, \mathcal{F}in)$-complex by assumption, the direct sums above have only finitely many nonzero summands. By [33, Corollary 9.7.2], we have a natural equivalence

$$HP(H(Z_g/\langle g \rangle, k[X^g])) \xrightarrow{\sim} \prod_{p\in\mathbb{Z}} H(Z_g/\langle g \rangle, k[X^g])[2p].$$

Summing up we have a natural quasi-isomorphism

$$\prod_{p\in\mathbb{Z}} \left(\bigoplus_{(g)\in \mathrm{con}_f G} H(Z_g, k[X^g]) \right) [2p] \xrightarrow{\sim} HP(k[\mathcal{G}^G(X)]/k) \xrightarrow{\sim} H^G(X, HP(k/k)). \tag{2.1.3}$$

Taking into account (2.1.2) we obtain a quasi-isomorphism of chain complexes

$$\bigoplus_{(g)\in \mathrm{con}_f G} H(Z_g, k[X^g]) \xrightarrow{\sim} H^G(X, HH(k/k)). \tag{2.1.4}$$

Moreover, in (2.1.3) we can replace $\prod_{p\in\mathbb{Z}}$ by $\bigoplus_{p\in\mathbb{Z}}$ because $H^G_n(X, HH(k/k)) = 0$ for $n \neq 0$. Indeed, X is a finite disjoint union of homogeneous spaces G/K with $K \in \mathcal{F}in$, and

$$H^G_n(G/K, HH(k/k)) = H^K_n(K/K, HH(k/k)) = HH_n(k[K]/k)$$

which is zero in positive dimensions since $k[K]$ is separable for finite K. This concludes the proof. □

Proposition 2.1.5. *(cf.* [27, Rmk. 1.9]*) If $k \supset \mathbb{Q}$ is a field, then the assembly map*

$$H^G_*(\mathcal{E}(G, \mathcal{F}in), HP(k/k)) \to HP_*(k[G]/k)$$

is injective.

Proof. The inclusion

$$C(k[G]/k) = \bigoplus_{(g)\in \mathrm{con}G} C^{(g)}(k[G]/k) \subset \prod_{(g)\in \mathrm{con}G} C^{(g)}(k[G]/k)$$

induces a chain map $HP(k[G]/k) \to \prod_{(g)\in \mathrm{con}G} HP^{(g)}(k[G]/k)$. Projecting onto the conjugacy classes of elements of finite order and taking homology we obtain a homomorphism

$$HP_n(k[G]/k) \to \prod_{(g)\in \mathrm{con}_f G} HP_n^{(g)}(k[G]/k). \tag{2.1.6}$$

Now for g of finite order $C^{(g)}(k[G]/k) = H(Z_g, k) \xrightarrow{\sim} H(Z_g/\langle g\rangle, k)$, hence by [33, Cor. 9.7.2]

$$HP_n^{(g)}(k[G]/k) = \prod_m H_{n+2m}(Z_g, k[pt]) = \prod_m H_{n+2m}(Z_g, k[\mathcal{E}(G, \mathcal{F}in)^g]).$$

One checks that the composite of the assembly map with the map (2.1.6) is the inclusion

$$\begin{aligned}
H_n^G(\mathcal{E}(G,\mathcal{F}in), HP(k/k)) &= \bigoplus_m H_{n+2m}^G(\mathcal{E}(G,\mathcal{F}in), HH(k/k)) \text{ (by Prop. 2.1.1)}\\
&\subset \prod_m H_{n+2m}^G(\mathcal{E}(G,\mathcal{F}in), HH(k/k))\\
&= \prod_m \bigoplus_{(g)\in \mathrm{con}_f G} H_{n+2m}(Z_g, k[\mathcal{E}(G,\mathcal{F}in)^g]) \text{ (by (2.1.4))}\\
&\subset \prod_m \prod_{(g)\in \mathrm{con}_f G} H_{n+2m}(Z_g, k[\mathcal{E}(G,\mathcal{F}in)^g]).
\end{aligned}$$

□

2.2 Equivariant Connes-Karoubi Chern character

In this section we consider algebras over a field k of characteristic zero. Recall from [6, §8.2] that the homotopy K-theory and periodic cyclic homology of a k-linear category are related by a Connes-Karoubi Chern character

$$KH(\mathcal{C}) \xrightarrow{ch} HP(\mathcal{C}/k)\ . \tag{2.2.1}$$

In particular if G is a group, $H \subset G$ a subgroup and R a unital k-algebra we have a map of OrG-spectra

$$ch : KH(R[\mathcal{G}^G(G/H)]) \to HP(R[\mathcal{G}^G(G/H)]/k). \tag{2.2.2}$$

By (1.3.6), Standing Assumptions 1.3.2 and Examples 1.3.3, this map is equivalent to the Chern character

$$ch : KH(R[H]) \to HP(R[H]/k) \tag{2.2.3}$$

for each fixed H. Using excision, all this extends to an arbitrary nonunital algebra A in place of R. We are interested in the particular case when k is either $\mathbb{C}$ or $\mathbb{Q}$ and $A \lhd \mathcal{B}$ is an ideal in the algebra $\mathcal{B}$ of bounded operators in a separable complex Hilbert space. Let $p > 0$; write $\mathcal{L}^p \lhd \mathcal{B}$ for the Schatten ideal of those compact operators whose sequence of singular values is p-summable. The operator trace $\mathrm{Tr} : \mathcal{L}^1 \to \mathbb{C}$ induces a map of cyclic modules

$$\begin{gathered}\mathrm{Tr} : C(\widetilde{\mathcal{L}^1}_{\mathbb{C}}[\mathcal{G}^G(G/H)] : \mathcal{L}^1[\mathcal{G}^G(G/H)]/\mathbb{C}) \to C(\mathbb{C}[\mathcal{G}^G(G/H)]/\mathbb{C}) \\ \mathrm{Tr}(a_0 \otimes g_0 \otimes \cdots \otimes a_n \otimes g_n) = \mathrm{Tr}(a_0 \cdots a_n) g_0 \otimes \cdots \otimes g_n.\end{gathered} \tag{2.2.4}$$

Note that Tr is defined on $a_0 \cdots a_n$ since at least one of the a_i is in $\mathcal{L}^1$. In particular Tr induces a natural transformation of OrG-chain complexes

$$\mathrm{Tr} : HP(\mathcal{L}^1[\mathcal{G}^G(G/H)]/\mathbb{C}) \to HP(\mathbb{C}[\mathcal{G}^G(G/H)]/\mathbb{C}).$$

Proposition 2.2.5. *Let X be a $(G, \mathcal{F}in)$-complex and $\mathcal{L}^1 \lhd \mathcal{B}$ the ideal of trace class operators. Then the composite*

$$c : H^G(X, KH(\mathcal{L}^1)) \otimes \mathbb{C} \xrightarrow{ch} H^G(X, HP(\mathcal{L}^1/\mathbb{C})) \xrightarrow{\mathrm{Tr}} H^G(X, HP(\mathbb{C}/\mathbb{C}))$$

is an equivalence.

Proof. It suffices to consider the case $X = G/H$ with $H \in \mathcal{F}in$. By (2.2.2), (2.2.3) and excision, we have a homotopy commutative diagram with vertical equivalences

$$\begin{tikzcd} KH(\mathcal{L}^1[H]) \otimes \mathbb{C} \arrow[r] \arrow[d, "\wr"] & HP(\mathcal{L}^1[H]/\mathbb{C}) \arrow[r] \arrow[d, "\wr"] & HP(\mathbb{C}[H]/\mathbb{C}) \arrow[d, "\wr"] \\ KH(\mathcal{L}^1[\mathcal{G}^G(G/H)]) \otimes \mathbb{C} \arrow[r] & HP(\mathcal{L}^1[\mathcal{G}^G(G/H)]/\mathbb{C}) \arrow[r] & HP(\mathbb{C}[\mathcal{G}^G(G/H)]/\mathbb{C}). \end{tikzcd}$$

Because $H \in \mathcal{F}in$, $\mathbb{C}[H]$ is Morita equivalent to its center, which is a sum of copies of $\mathbb{C}$ indexed by the conjugacy classes of H:

$$\mathbb{C}[H] \sim Z(\mathbb{C}[H]) = \bigoplus_{\mathrm{con}(H)} \mathbb{C}. \tag{2.2.6}$$

Since the (periodic) cyclic homology of $\mathbb{C}$ as a $\mathbb{C}$-algebra and as a locally convex topological algebra agree, it follows that the map $HP(\mathbb{C}[H]/\mathbb{C}) \to HP^{\mathrm{top}}(\mathbb{C}[H])$ is the identity. Next recall from [10, Notation 5.1 and Theorem 6.2.1 (iii)] that if $I \lhd \mathcal{B}$ is an operator ideal with a Banach algebra structure such that the inclusion $I \subset \mathcal{B}$ is continuous and $\otimes_\pi$ is the projective tensor product (see Appendix A.1), then for every locally convex algebra A the comparison map $KH(I \otimes_\pi A) \to K^{\mathrm{top}}(I \otimes_\pi A)$ is an equivalence. In particular this applies when $I = \mathcal{L}^1$ and $A = \mathbb{C}[H]$. Hence we have a homotopy commutative diagram with vertical equivalences

$$\begin{tikzcd} KH(\mathcal{L}^1[H]) \otimes \mathbb{C} \arrow[rr, "c"] \arrow[d, "\wr"] & & HP(\mathbb{C}[H]/\mathbb{C}) \arrow[d, "\wr"] \\ K^{\mathrm{top}}(\mathcal{L}^1[H]) \otimes \mathbb{C} \arrow[r, "ch^{\mathrm{top}}"'] & HP^{\mathrm{top}}(\mathcal{L}^1[H]) \arrow[r, "\mathrm{Tr}"'] & HP^{\mathrm{top}}(\mathbb{C}[H]). \end{tikzcd}$$

Here ch^{top} is the topological Connes-Karoubi Chern character. Using (2.2.6) we have that Tr is an equivalence by [11, Prop. 17.3] and ch^{top} is an equivalence because of (2.2.6) and of the commutativity of the following diagram

$$\begin{tikzcd} K^{\mathrm{top}}(\mathcal{L}^1)\otimes\mathbb{C} \arrow[r, "ch^{\mathrm{top}}"] & HP^{\mathrm{top}}(\mathcal{L}^1/\mathbb{C}) \\ K^{\mathrm{top}}(\mathbb{C})\otimes\mathbb{C} \arrow[u, "\wr"] \arrow[r, "ch^{\mathrm{top}}", "\sim"'] & HP^{\mathrm{top}}(\mathbb{C}/\mathbb{C}). \arrow[u, "\wr"] \end{tikzcd}$$

It follows that c is an equivalence. This concludes the proof. □

Corollary 2.2.7. *Let X be a $(G,\mathcal{F}in)$ complex. Then, for every $p>0$ there is an equivalence*

$$c : H^G(X, KH(\mathcal{L}^p))\otimes\mathbb{C}\to H^G(X, HP(\mathbb{C}/\mathbb{C})).$$

Proof. Because $\mathcal{L}^1/\mathcal{L}^p$ for $p<1$ and $\mathcal{L}^p/\mathcal{L}^1$ for $p>1$ are nilpotent rings and KH is nilinvariant ([34]), the maps

$$KH(\mathcal{L}^p[\mathcal{G}^G(-)])\to KH(\mathcal{L}^1[\mathcal{G}^G(-)])\quad (p<1)$$

and

$$KH(\mathcal{L}^1[\mathcal{G}^G(-)])\to KH(\mathcal{L}^p[\mathcal{G}^G(-)])\quad (p>1)$$

are equivalences of OrG-spectra by Remark 1.3.5 and (1.3.10). The proof is now immediate from Proposition 2.2.5. □

2.3 The KH-assembly map with $\mathcal{L}^p$-coefficients

Theorem 2.3.1. *Let $p>0$ and G a group. Then the rational assembly map*

$$H^G_*(\mathcal{E}(G,\mathcal{F}in), KH(\mathcal{L}^p))\otimes\mathbb{Q}\to KH_*(\mathcal{L}^p[G])\otimes\mathbb{Q}$$

is injective.

Proof. It suffices to show that the map tensored with $\mathbb{C}$ is injective. We have a commutative diagram

$$\begin{tikzcd} H^G_*(\mathcal{E}(G,\mathcal{F}in), KH(\mathcal{L}^p))\otimes\mathbb{C} \arrow[r] \arrow[d, "c"', "\wr"] & KH_*(\mathcal{L}^p[G])\otimes\mathbb{C} \arrow[d, "c"] \\ H^G_*(\mathcal{E}(G,\mathcal{F}in), HP(\mathbb{C}/\mathbb{C})) \arrow[r, tail] & HP_*(\mathbb{C}[G]/\mathbb{C}). \end{tikzcd}$$

The vertical map on the left is an isomorphism by Corollary 2.2.7; the bottom horizontal map is injective by Proposition 2.1.5. It follows that the top horizontal map is injective. This concludes the proof. □

Let $\mathcal{S}=\bigcup_{p>0}\mathcal{L}^p$ be the ring of all Schatten operators. Because $\mathcal{S}^2=\mathcal{S}$, the ring $\mathcal{S}$ is K-excisive by [31, Thm. C] and [10, Proof of Thm. 8.2.1] (see also [35, Thm. 4]). We can now deduce the following result of Guoliang Yu.

Corollary 2.3.2. *(*[36, Thm. 1.1]*). The rational assembly map*

$$H_*^G(\mathcal{E}(G,\mathcal{V}cyc),K(\mathcal{S}))\otimes\mathbb{Q}\to K_*(\mathcal{S}[G])\otimes\mathbb{Q}$$

is injective.

Proof. Because $\mathcal{S}=\mathcal{S}^2$ and the tensor product of operators $\mathcal{B}\otimes\mathcal{B}\to\mathcal{B}(\mathcal{H}\otimes\mathcal{H})\cong\mathcal{B}$ (see Appendix A.2.1) sends $\mathcal{L}^1\otimes\mathcal{S}\to\mathcal{S}$, the operator ideal $\mathcal{S}$ is sub-harmonic in the sense of [10, Def. 6.5.1]. Hence the map $K(A\otimes_{\mathbb{C}}\mathcal{S})\to KH(A\otimes_{\mathbb{C}}\mathcal{S})$ is an equivalence for every H-unital $\mathbb{C}$-algebra A ([10, Thm. 8.2.5, Rmk. 8.2.6]). Applying this when $A=\mathbb{C}[H]$ and using the fact that both K and KH satisfy 1.3.2, we obtain an equivalence of OrG-spectra $K(\mathcal{S}[\mathcal{G}^G(G/H)])\to KH(\mathcal{S}[\mathcal{G}^G(G/H)])$ (see Remark 1.3.5). Hence for every G-simplicial set X we have a homotopy commutative diagram with vertical equivalences

$$\begin{tikzcd} H^G(X,K(\mathcal{S})) \arrow[r, "assem"] \arrow[d, "\wr"] & K(\mathcal{S}[G]) \arrow[d, "\wr"] \\ H^G(X,KH(\mathcal{S})) \arrow[r, "assem"] & KH(\mathcal{S}[G]). \end{tikzcd} \tag{2.3.3}$$

On the other hand by [2, Thm. 7.4] and (1.3.11), for every ring A the map

$$\mathcal{E}(G,\mathcal{F}in)\to\mathcal{E}(G,\mathcal{V}cyc)$$

induces a weak equivalence

$$H^G(\mathcal{E}(G,\mathcal{F}in),KH(A))\xrightarrow{\sim}H^G(\mathcal{E}(G,\mathcal{V}cyc),KH(A)).$$

Applying this to $A=\mathcal{S}$ and using Theorem 2.3.1 we obtain that when $X=\mathcal{E}(G,\mathcal{V}cyc)$ the bottom horizontal arrow in (2.3.3) –and thus also the top arrow– is a rational equivalence. □

2.4 Resumen

Sea $\mathcal{B}$ el anillo de operadores acotados en un espacio de Hilbert de dimensión infinita, complejo y separable. Recordemos que los *valores singulares* de un operador acotado $T\in\mathcal{B}$ son las raíces cuadradas de los autovalores del operador autoadjunto y positivo T^*T. Sea $p>0$; llamamos $\mathcal{L}^p\lhd\mathcal{B}$ al ideal de Schatten formado por los operadores compactos cuya sucesión de valores singulares es p-sumable. Sea $\mathcal{S}=\bigcup_{p>0}\mathcal{L}^p$ el anillo de todos los operadores de Schatten. Por [10, Thm. 8.2.1] (ver también [35, Thm. 4]), $\mathcal{S}$ es K-escisivo. Luego tiene sentido considerar el mapa de ensamble $(2')$ con coeficientes en $K(\mathcal{S})$; Guoliang Yu probó en [36] que es racionalmente inyectivo. Su demostración se basa en la construcción de un caracter de Chern que funciona para el caso particular con coeficientes $K(\mathcal{S})$, y en resultados sobre K-teoría algebraica de ideales de operadores ([10], [35]), y sobre topología controlada y geometría de escala ([1], [29]).

En este capítulo damos una prueba diferente del resultado de Yu. En nuestra demostración usamos el caracter de Chern usual con valores en la homología cíclica. Al igual que Yu, nos basamos en resultados sobre K-teoría algebraica de ideales de operadores, pero no utilizamos herramientas de topología controlada ni geometría de escala. Formulamos el resultado en términos de KH y probamos:

Teorema 2.4.1. *Sean $p > 0$ y G un grupo. El morfismo de ensamble racional*

$$H_*^G(\mathcal{E}(G, \mathcal{F}in), KH(\mathcal{L}^p)) \otimes \mathbb{Q} \to KH_*(\mathcal{L}^p[G]) \otimes \mathbb{Q}$$

es inyectivo.

El resultado de Yu se obtiene como corolario.

Corolario 2.4.2. *(*[36, Thm. 1.1]*). Sea G un grupo y sea $\mathcal{S} = \bigcup_{p>0} \mathcal{L}^p$ el anillo de todos los operadores de Schatten. Entonces el morfismo de ensamble racional*

$$H_*^G(\mathcal{E}(G, \mathcal{V}cyc), K(\mathcal{S})) \otimes \mathbb{Q} \to K_*(\mathcal{S}[G]) \otimes \mathbb{Q}$$

es inyectivo.

La prueba del corolario deja en claro que los dos morfismos de ensamble son isomorfos, por lo que realmente es el mismo resultado.

El resto de este capítulo está organizado de la siguiente manera. En la Sección 2.1 mostramos que si X es un $(G, \mathcal{F}in)$-complejo y $k \supset \mathbb{Q}$ es un cuerpo, entonces

$$H_n^G(X, HP(k/k)) = \bigoplus_{p \in \mathbb{Z}} H_{n+2p}^G(X, HH(k/k)).$$

Usamos esto para mostrar en la Proposición 2.1.5 que para cualquier grupo G, el morfismo de ensamble

$$H_n^G(\mathcal{E}(G, \mathcal{F}in), HP(k/k)) \to HP_n(k[G]/k) \tag{2.4.3}$$

es inyectivo. En la Sección 2.2 consideramos el caracter de Chern de Connes-Karoubi

$$ch : H^G(X, KH(A)) \to H^G(X, HP(A/k))$$

definido en [6, §8]. En la Proposición 2.2.5 mostramos que la composición de ch con el operador de traza resulta una equivalencia

$$c : H^G(X, KH(\mathcal{L}^1)) \otimes \mathbb{C} \xrightarrow{\sim} H^G(X, HP(\mathbb{C}/\mathbb{C}))$$

para todo $(G, \mathcal{F}in)$-complejo X. De esto, y del hecho que $KH_*(\mathcal{L}^1) \cong KH_*(\mathcal{L}^p)$ deducimos –en el Corolario 2.2.7– que para todo $p > 0$ se obtiene una equivalencia similar, a la que también llamamos c

$$c : H^G(X, KH(\mathcal{L}^p)) \otimes \mathbb{C} \xrightarrow{\sim} H^G(X, HP(\mathbb{C}/\mathbb{C})). \tag{2.4.4}$$

En la Sección 2.3 demostramos el Teorema 2.4.1, usando (2.4.4) y la inyectividad de (2.4.3). El resultado de Yu se prueba en el Corolario 2.3.2.

Estos resultados fueron publicados en [7].

3. TRACE CLASS OPERATORS, REGULATORS AND ASSEMBLY MAPS

Let G be a group, $\mathcal{F}in$ the family of its finite subgroups, and $\mathcal{L}^1$ the algebra of trace-class operators in an infinite dimensional, separable Hilbert space over the complex numbers. In the previous chapter we proved that the rational assembly map in homotopy algebraic K-theory

$$H_p^G(\mathcal{E}(G,\mathcal{F}in), KH(\mathcal{L}^1)) \otimes \mathbb{Q} \to KH_p(\mathcal{L}^1[G]) \otimes \mathbb{Q} \tag{3.1}$$

is injective. The rational KH-isomorphism conjecture ([2, Conjecture 7.3]) predicts that (3.1) is an isomorphism. In this chapter we prove the following

Theorem 3.2. *Assume that* (3.1) *is surjective. Let* $n \equiv p+1 \mod 2$. *Then:*

i) The rational assembly map for the trivial family

$$H_n^G(\mathcal{E}(G,\{1\}), K(\mathbb{Z})) \otimes \mathbb{Q} \to K_n(\mathbb{Z}[G]) \otimes \mathbb{Q} \tag{3.3}$$

is injective.

ii) For every number field F, the rational assembly map

$$H_n^G(\mathcal{E}(G,\mathcal{F}in), K(F)) \otimes \mathbb{Q} \to K_n(F[G]) \otimes \mathbb{Q} \tag{3.4}$$

is injective.

We remark that the K-theory Novikov conjecture asserts that part i) of the theorem above holds for all G, and that part ii) is equivalent to the rational injectivity part of the K-theory Farrell-Jones conjecture for number fields ([26, Conjectures 51 and 58 and Proposition 70]).

The idea of the proof of Theorem 3.2 is to use an algebraic, equivariant version of Karoubi's multiplicative K-theory. The latter theory assigns groups $MK_n(\mathfrak{A})$ $(n \geq 1)$ to any unital Banach algebra $\mathfrak{A}$, which fit into a long exact sequence

$$HC_{n-1}^{\mathrm{top}}(\mathfrak{A}) \to MK_n(\mathfrak{A}) \to K_n^{\mathrm{top}}(\mathfrak{A}) \overset{Sch_n^{\mathrm{top}}}{\to} HC_{n-2}^{\mathrm{top}}(\mathfrak{A}).$$

Here HC^{top} is the cyclic homology of the completed cyclic module $C_n^{\mathrm{top}}(\mathfrak{A}) = \mathfrak{A} \otimes_\pi \ldots \otimes_\pi \mathfrak{A}$ $(n+1$ factors), ch^{top} is the Connes-Karoubi Chern character with values in its periodic cyclic homology $HP_*^{\mathrm{top}}(\mathfrak{A})$, and S is the periodicity operator. Karoubi introduced a multiplicative Chern character

$$\mu_n : K_n(\mathfrak{A}) \to MK_n(\mathfrak{A}). \tag{3.5}$$

In particular if $\mathcal{O}$ is the ring of integers in a number field F one can consider the composite

$$K_n(\mathcal{O}) \to K_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^{\hom(F,\mathbb{C})}. \tag{3.6}$$

By comparing this map with the Borel regulator, Karoubi showed in [22] that (3.6) is rationally injective. It follows that

$$K_n(\mathbb{Z}) \to MK_n(\mathbb{C}) \tag{3.7}$$

is rationally injective. In this chapter we assign, to every unital $\mathbb{C}$-algebra A, groups $\kappa_n(A)$ ($n \in \mathbb{Z}$) which fit into a long exact sequence

$$HC_{n-1}(A/\mathbb{C}) \to \kappa_n(A) \to KH_n(\mathcal{L}^1 \otimes_{\mathbb{C}} A) \overset{\mathrm{Tr}\mathcal{S}ch_n}{\to} HC_{n-2}(A/\mathbb{C}).$$

Here $HC(/\mathbb{C})$ is algebraic cyclic homology of $\mathbb{C}$-algebras, ch is the algebraic Connes-Karoubi Chern character and Tr is induced by the operator trace. We also introduce a character

$$\tau_n : K_n(A) \to \kappa_n(A) \quad (n \in \mathbb{Z}). \tag{3.8}$$

If $\mathfrak{A}$ is a finite dimensional Banach algebra and $n \geq 1$ then $\kappa_n(\mathfrak{A}) = MK_n(\mathfrak{A})$ and (3.5) identifies with (3.8) (Proposition 3.1.2.2). Both κ and τ have equivariant versions, so that if X is a G-space and A is a $\mathbb{C}$-algebra, we have an assembly map

$$H_n^G(X, \kappa(A)) \to \kappa_n(A[G]).$$

Let $\mathcal{F}cyc$ be the family of finite cyclic subgroups. We show in Proposition 3.2.4 that the map

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \to H_n^G(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C}))$$

is an isomorphism, and compute $H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C}))\otimes\mathbb{Q}$ in terms of the finite cyclic subgroups of G. We use this and the rational injectivity of (3.7) to show, in Proposition 3.4.2, that the map

$$H_n^G(\mathcal{E}(G, \{1\}), K(\mathbb{Z})) \to H_n^G(\mathcal{E}(G, \mathcal{F}in), K(\mathbb{C})) \overset{\tau}{\to} H_n^G(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C})) \tag{3.9}$$

is rationally injective. It is well-known [26, Proposition 76] that the map

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), K(R)) \otimes \mathbb{Q} \to H_n^G(\mathcal{E}(G, \mathcal{F}in), K(R)) \otimes \mathbb{Q}$$

is an isomorphism for every unital ring R. In particular, we may substitute $\mathcal{F}cyc$ for $\mathcal{F}in$ in (3.4). We use this together with Proposition 3.2.4 and the rational injectivity of

$$K_n(F) \to K_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^{\hom(F,\mathbb{C})}$$

(see Remark 3.1.3.2), to show in Proposition 3.4.6 that if $m \geq 1$, $\mathcal{C}yc_m$ is the family of cyclic subgroups whose order divides m, and ζ_m is a primitive m-root of 1, then the composite

$$\begin{tikzcd}
H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), K(F)) \otimes \mathbb{Q} \arrow[r] \arrow[dr, dotted] & H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), K(\mathbb{C}))^{\hom(F(\zeta_m),\mathbb{C})} \otimes \mathbb{Q} \arrow[d, "\tau"] \\
 & H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C}))^{\hom(F(\zeta_m),\mathbb{C})} \otimes \mathbb{Q}
\end{tikzcd} \tag{3.10}$$

is injective. Since the map $\operatorname{colim}_m \mathcal{E}(G, \mathcal{C}yc_m) \to \mathcal{E}(G, \mathcal{F}cyc)$ is an equivalence, it follows that if the rational assembly map

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q} \to \kappa_n(\mathbb{C}[G]) \otimes \mathbb{Q} \tag{3.11}$$

is injective then so are both (3.3) and (3.4). We show in Corollary 3.3.7 that if (3.1) is surjective, then (3.11) is injective for $n \equiv p+1 \mod 2$. This proves Theorem 3.2.

The rest of this chapter is organized as follows. In Section 3.1 we define $\kappa_n(A)$ and the map $\tau_n : K_n(A) \to \kappa_n(A)$. By definition, if $n \leq 0$, then $\kappa_n(A) = KH_n(A \otimes_{\mathbb{C}} \mathcal{L}^1)$ and τ_n is the identity map (3.1.1.3). We show in Proposition 3.1.2.2 that if $n \geq 1$ and $\mathfrak{A}$ is a finite dimensional Banach algebra, then $\kappa_n(\mathfrak{A}) = MK_n(\mathfrak{A})$ and $\tau_n = \mu_n$. Karoubi's regulators and his injectivity results are recalled in Theorem 3.1.3.1. We use Karoubi's theorem to prove, in Lemma 3.1.3.4, that if F is a number field, C a cyclic group of order m, n a multiple of m, and ζ_n a primitive n-root of 1, then the composite

$$K_*(F[C]) \to K_*(F(\zeta_n)[C]) \to K_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})} \xrightarrow{\mu} MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}$$

is rationally injective. The main result of Section 3.2 is Proposition 3.2.4, which computes $H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q}$ in terms of group homology and of the groups $\kappa_*(\mathbb{C}[C])$ for $C \in \mathcal{F}cyc$. The resulting formula is similar to existing formulas for equivariant K and cyclic homology, which are used in its proof ([6], [24], [25], [26],[27]). In Section 3.3 we show that the rational $\kappa(\mathbb{C})$-assembly map is injective whenever the rational $KH(\mathcal{L}^1)$-assembly map is surjective (Corollary 3.3.7). For this we use the fact that for every $m \geq 1$, the assembly map

$$H_*^G(\mathcal{E}(G, \mathcal{C}yc_m), HC(\mathbb{C}/\mathbb{C})) \to HC_*(\mathbb{C}[G])$$

has a natural left inverse π_m, which makes the following diagram commute

$$\begin{tikzcd}
H_*^G(\mathcal{E}(G, \mathcal{C}yc_m), KH(\mathcal{L}^1)) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\mathrm{Tr}Sch"] & KH_*(\mathcal{L}^1[G]) \otimes \mathbb{Q} \arrow[d, "\mathrm{Tr}Sch"] \\
H_{*-2}^G(\mathcal{E}(G, \mathcal{C}yc_m), HC(\mathbb{C}/\mathbb{C})) & HC_{*-2}(\mathbb{C}[G]). \arrow[l, "\pi_m"']
\end{tikzcd}$$

Hence for every n we have an inclusion

$$\mathrm{Tr}Sch(H_{n+1}^G(\mathcal{E}(G, \mathcal{C}yc_m), KH(\mathcal{L}^1)) \otimes \mathbb{Q}) \subset \pi_m \mathrm{Tr}Sch(KH_{n+1}(\mathcal{L}^1[G]) \otimes \mathbb{Q}). \tag{3.12}$$

We show in Proposition 3.3.5 that the rational assembly map

$$H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), \kappa(\mathbb{C})) \otimes \mathbb{Q} \to \kappa_n(\mathbb{C}[G]) \otimes \mathbb{Q}$$

is injective if and only if the inclusion (3.12) is an equality. Corollary 3.3.7 is immediate from this. Section 3.4 is concerned with proving that (3.9) and (3.10) are injective (Propositions 3.4.2 and 3.4.6). Finally in Section 3.5 we show that if the identity holds in (3.12) for $m = 1$ then (3.3) is injective (Theorem 3.5.1) and that if it holds for m, then

$$H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), K(F)) \otimes \mathbb{Q} \to K_n(F[G]) \otimes \mathbb{Q}$$

is injective for every number field F (Theorem 3.5.2).

These results were published in [8].

3.1 *The character $\tau : K(A) \to \kappa(A)$*

3.1.1 *Definition of τ*

Let A be a $\mathbb{C}$-algebra and $k \subset \mathbb{C}$ a subfield. When $k = \mathbb{Q}$ we omit it from our notation; thus for example, $HH(A) = HH(A/\mathbb{Q})$. As usual, we write S, B and I for the maps appearing in Connes' SBI sequence. We write K^{nil} for the fiber of the comparison map $K \to KH$. We have a map of fibration sequences [5, §11.3]

$$\begin{tikzcd} K^{\mathrm{nil}}(A) \arrow[r] \arrow[d, "\nu"'] & K(A) \arrow[r] \arrow[d] & KH(A) \arrow[d, "ch"] \\ HC(A)[-1] \arrow[r, "B"'] & HN(A) \arrow[r, "I"'] & HP(A). \end{tikzcd}$$

We shall abuse notation and write Sch for the map that makes the following diagram commute

$$\begin{tikzcd} KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \arrow[r, "ch"] \arrow[d, "Sch"'] & HP(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \arrow[d, "\wr"] \\ HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[-1] & HP(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1]. \arrow[l, "S"] \end{tikzcd} \tag{3.1.1.1}$$

By [10, Theorems 6.5.3 and 7.1.1], the map

$$\nu : K^{nil}(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[-1]$$

is an equivalence, and thus the map Sch fits into a fibration sequence

$$KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \xrightarrow{Sch} HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[-1] \to K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A).$$

On the other hand, we have the chain map induced by the operator trace $\mathrm{Tr} : \mathcal{L}^1 \to \mathbb{C}$ (see (2.2.4))

$$\mathrm{Tr} : HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to HC(A/\mathbb{C}). \tag{3.1.1.2}$$

We define $\kappa(A)$ as the homotopy cofiber of the composite of (3.1.1.2) and the map Sch of (3.1.1.1)

$$\kappa(A) := \mathrm{hocofiber}(KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \xrightarrow{\mathrm{Tr}Sch} HC(A/\mathbb{C})[-1]).$$

Thus because, by definition, cyclic homology vanishes in negative degrees, we have

$$\kappa_n(A) = KH_n(\mathcal{L}^1 \otimes_{\mathbb{C}} A) \quad (n \leq 0). \tag{3.1.1.3}$$

By construction, there is an induced map $K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to \kappa(A)$ which fits into a commutative diagram

$$\begin{tikzcd} KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \arrow[r, "Sch"] \arrow[d, equal] & HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)[-1] \arrow[r] \arrow[d, "Tr"] & K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \arrow[d] \\ KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A)[+1] \arrow[r, "TrSch"'] & HC(A/\mathbb{C})[-1] \arrow[r] & \kappa(A). \end{tikzcd} \tag{3.1.1.4}$$

A choice of a rank one projection p gives a map $A \to \mathcal{L}^1 \otimes_{\mathbb{C}} A$, $a \mapsto p \otimes a$, and therefore a map $K(A) \to K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)$. We shall be interested in the composite

$$\tau : K(A) \to K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A) \to \kappa(A). \tag{3.1.1.5}$$

3.1.2 Comparison with Karoubi's multiplicative Chern character

Suppose now that $\mathfrak{A}$ is a unital Banach algebra. Let $\Delta^{\mathrm{diff}}_{\bullet}\mathfrak{A} = C^{\infty}(\Delta_{\bullet}, \mathfrak{A})$ be the simplicial algebra of $\mathfrak{A}$-valued C^{∞}-functions on the standard simplices. Write $KV^{\mathrm{diff}}(\mathfrak{A})$ for the diagonal of the bisimplicial space $[n] \mapsto BGL(\Delta^{\mathrm{diff}}_n\mathfrak{A})$. We have

$$K^{\mathrm{top}}_n(\mathfrak{A}) = \pi_n KV^{\mathrm{diff}}(\mathfrak{A}) \ \ (n \geq 1).$$

Consider the fiber $\mathcal{F}(\mathfrak{A}) = \mathrm{hofiber}(BGL^{+}(\mathfrak{A}) \to KV^{\mathrm{diff}}(\mathfrak{A}))$. We have a homotopy fibration

$$\Omega BGL^{+}(\mathfrak{A}) \to \Omega KV^{\mathrm{diff}}(\mathfrak{A}) \to \mathcal{F}(\mathfrak{A}). \tag{3.1.2.1}$$

Let $\otimes_{\pi}$ be the projective tensor product of Banach spaces and let $C^{\mathrm{top}}(\mathfrak{A})$ be the cyclic module with $C^{\mathrm{top}}(\mathfrak{A})_n = \mathfrak{A} \otimes_{\pi} \ldots \otimes_{\pi} \mathfrak{A}$ ($n+1$ factors). Write $HC^{\mathrm{top}}(\mathfrak{A})$ and $HP^{\mathrm{top}}(\mathfrak{A})$ for the cyclic and periodic cyclic complexes of $C^{\mathrm{top}}(\mathfrak{A})$. In [23] (see also [22, §7]), Max Karoubi constructs a map $ch^{\mathrm{rel}} : \mathcal{F}(\mathfrak{A}) \to HC^{\mathrm{top}}(\mathfrak{A})[-1]$ and defines his multiplicative K-groups as the homotopy groups

$$MK_n(\mathfrak{A}) = \pi_n(\mathrm{hofiber}(KV^{\mathrm{diff}}(\mathfrak{A}) \to HC^{\mathrm{top}}(\mathfrak{A})[-2])) \quad (n \geq 1).$$

He further defines the multiplicative Chern character as the induced map $\mu_n : K_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ $(n \geq 1)$.

Proposition 3.1.2.2. *Let $\mathfrak{A}$ be a unital Banach algebra, and let $n \geq 1$. Then there is a natural map $\kappa_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ which makes the following diagram commute*

$$\begin{tikzcd} K_n(\mathfrak{A}) \arrow[r, "\tau_n"] \arrow[dr, "\mu_n"'] & \kappa_n(\mathfrak{A}) \arrow[d] \\ & MK_n(\mathfrak{A}). \end{tikzcd}$$

If furthermore $\mathfrak{A}$ is finite dimensional, then $\kappa_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ is an isomorphism.

Proof. Consider the simplicial ring

$$\Delta_\bullet\mathfrak{A} : [n] \mapsto \Delta_n\mathfrak{A} = \mathfrak{A}[t_0, \dots, t_n]/\langle 1 - \sum_{i=0}^{n} t_i\rangle.$$

Let $KV(\mathfrak{A})$ be the diagonal of the bisimplicial set $BGL(\Delta_\bullet\mathfrak{A})$. We have a homotopy commutative diagram

$$\begin{tikzcd}
KV(\mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A}) \arrow[r] \arrow[d] & KV^{\mathrm{diff}}(\mathcal{L}^1 \otimes_\pi \mathfrak{A}) \arrow[d] & KV^{\mathrm{diff}}(\mathfrak{A}) \arrow[l] \arrow[d] \\
HC(\mathcal{B} \otimes_{\mathbb{C}} \mathfrak{A} : \mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A})[-2] \arrow[r] \arrow[d, "\mathrm{Tr}"] & HC^{\mathrm{top}}(\mathcal{B} \otimes_\pi \mathfrak{A} : \mathcal{L}^1 \otimes_\pi \mathfrak{A})[-2] \arrow[d] & HC^{\mathrm{top}}(\mathfrak{A})[-2] \arrow[l] \arrow[dl, equal] \\
HC(\mathfrak{A}/\mathbb{C})[-2] \arrow[r] & HC^{\mathrm{top}}(\mathfrak{A})[-2] &
\end{tikzcd} \tag{3.1.2.3}$$

By [10, Lemma 3.2.1 and Theorem 6.5.3(i)] and [34, Proposition 1.5] (or [5, Proposition 5.2.3]), the natural map

$$KV_n(\mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A}) \to KH_n(\mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A})$$

is an isomorphism for $n \geq 1$. It follows from this that for $n \geq 1$, the group $\kappa_n(\mathfrak{A})$ is isomorphic to π_n of the fiber of the composite of the first column of diagram (3.1.2.3). On the other hand, by Karoubi's density theorem, the map $KV^{\mathrm{diff}}(\mathfrak{A}) \to KV^{\mathrm{diff}}(\mathcal{L}^1 \otimes_\pi \mathfrak{A})$ is an equivalence; inverting it and taking fibers and homotopy groups, we get a natural map $\kappa_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ $(n \geq 1)$. The commutativity of the diagram of the proposition is clear. If now $\mathfrak{A}$ is finite dimensional, then $\mathfrak{A} \otimes_\pi V = \mathfrak{A} \otimes_{\mathbb{C}} V$ for any locally convex vector space V. Hence the map $HC(\mathfrak{A}/\mathbb{C}) \to HC^{\mathrm{top}}(\mathfrak{A})$ is the identity map. Furthermore, by [10, Theorem 3.2.1], the map $KV(\mathcal{L}^1 \otimes_{\mathbb{C}} \mathfrak{A}) \to KV^{\mathrm{diff}}(\mathcal{L}^1 \otimes_\pi \mathfrak{A})$ is an equivalence. It follows that $\kappa_n(\mathfrak{A}) \to MK_n(\mathfrak{A})$ is an isomorphism for all $n \geq 1$, finishing the proof. $\square$

Example 3.1.2.4. We have

$$\kappa_n(\mathbb{C}) = \begin{cases} \mathbb{C}^* & n \geq 1, \text{ odd.} \\ \mathbb{Z} & n \leq 0, \text{ even.} \\ 0 & \text{otherwise.} \end{cases}$$

3.1.3 Regulators

In view of Proposition 3.1.2.2 above we may substitute τ for μ in the theorem below.

Theorem 3.1.3.1. [22, Théorème 7.20] *Let $\mathcal{O}$ be the ring of integers in a number field F. Write $F \otimes \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$; put $r = r_1 + r_2$. Then the inclusion $\mathcal{O} \subset \mathbb{C}^r$ followed by the the map $\mu_n : K_n(\mathbb{C})^r \to MK_n(\mathbb{C})^r$ induces a monomorphism $K_n(\mathcal{O}) \otimes \mathbb{Q} \to MK_n(\mathbb{C})^r \otimes \mathbb{Q}$ $(n \geq 1)$.*

Remark 3.1.3.2. It follows from classical results of Quillen that the map $K_n(\mathcal{O}) \to K_n(F)$ is a rational isomorphism for $n \geq 2$. Thus $K_n(F) \to MK_n(\mathbb{C})^r$ is rationally injective for $n \geq 2$.

Moreover, $K_1(F) \to MK_1(\mathbb{C})^r$ is injective too, since the map $\tau_1 : K_1(\mathbb{C}) \to MK_1(\mathbb{C})$ is the identity of $\mathbb{C}^*$. Observe that the isomorphism $F \otimes \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ of Theorem 3.1.3.1 is not canonical; it implies choosing r_2 nonreal embeddings $F \to \mathbb{C}$ out of the total $2r_2$, so that no two of them differ by complex conjugation. On the other hand the map

$$\iota : F \to \mathbb{C}^{\hom(F,\mathbb{C})}, \quad \iota(x)_\sigma = \sigma(x)$$

is canonical. Moreover, the composite

$$\operatorname{reg}_n(F) : K_n(F) \to K_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^{\hom(F,\mathbb{C})} \tag{3.1.3.3}$$

is still a rational monomorphism. Indeed the map of the theorem is obtained by composing (3.1.3.3) with a projection $MK_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^r$.

Lemma 3.1.3.4. *Let F be a number field, C a cyclic group of order m, n a multiple of m and ζ_n a primitive n-th root of 1. Then the composite map*

$$K_*(F[C]) \to K_*(F(\zeta_n)[C]) \xrightarrow{\iota} K_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})} \xrightarrow{\mu} MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}$$

is rationally injective.

Proof. Let G_m be the Galois group of the extension of fields $F \subset F(\zeta_m)$; if M is a G_m-module, write M^{G_m} for the fixed points. By [24, Lemma 8.4], the map $F[C] \to F(\zeta_m)[C]$ induces an isomorphism $K_*(F[C]) \otimes \mathbb{Q} \to K_*(F(\zeta_m)[C])^{G_m} \otimes \mathbb{Q}$. In particular, $K_*(F[C]) \to K_*(F(\zeta_m)[C])$ is rationally injective. Now if $\sigma : F(\zeta_m) \to E$ is a field homomorphism, then $K_*(E[C]) = K_*(E)^m$, and the map $K_*(F(\zeta_m)[C]) \to K_*(E[C])$ decomposes into a direct sum of m copies of the map $K_*(F(\zeta_m)) \to K_*(E)$. In particular this applies when $E \in \{F(\zeta_n), \mathbb{C}\}$. In view of Theorem 3.1.3.1 and Remark 3.1.3.2, it follows that both $K_*(F(\zeta_m)[C]) \to MK_*(\mathbb{C}[C])^{\hom(F(\zeta_m),\mathbb{C})}$ and $K_*(F(\zeta_n)[C]) \to MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}$ are rationally injective. Summing up, we have a commutative diagram

$$\begin{tikzcd}
K_*(F[C]) \arrow[d] & \\
K_*(F(\zeta_m)[C]) \arrow[r] \arrow[d] & MK_*(\mathbb{C}[C])^{\hom(F(\zeta_m),\mathbb{C})} \arrow[d] \\
K_*(F(\zeta_n)[C]) \arrow[r] & MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}.
\end{tikzcd}$$

We have shown that the first vertical map on the left and the two horizontal maps are rationally injective. Since the vertical map on the right is injective, we conclude that the composite of the left column followed by the bottom horizontal arrow is a rational monomorphism, finishing the proof. ☐

3.2 Rational computation of equivariant κ-homology

Let G be a group. It follows from **§2.2**, that the diagram (3.1.1.4) can be promoted to a commutative diagram of OrG-spectra whose columns are homotopy fibrations

$$\begin{tikzcd}
KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)])[+1] \arrow[r, equal] \arrow[d, "Sch"] & KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)])[+1] \arrow[d, "\mathrm{Tr}Sch"] \\
HC(\mathcal{B} \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)] : \mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)])[-1] \arrow[r, "\mathrm{Tr}"] \arrow[d] & HC(A[\mathcal{G}(G/H)]/\mathbb{C})[-1] \arrow[d] \\
K(\mathcal{B} \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)] : \mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)]) \arrow[r] & \kappa(A[\mathcal{G}(G/H)]).
\end{tikzcd}$$

If now X is any G-simplicial set, then taking G-equivariant homology yields a diagram whose columns are again homotopy fibrations

$$\begin{tikzcd}
H^G(X, KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A))[+1] \arrow[r, equal] \arrow[d, "Sch"] & H^G(X, KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A))[+1] \arrow[d, "\mathrm{Tr}Sch"] \\
H^G(X, HC(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A))[-1] \arrow[r, "\mathrm{Tr}"] \arrow[d] & H^G(X, HC(A/\mathbb{C}))[-1] \arrow[d] \\
H^G(X, K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)) \arrow[r] & H^G(X, \kappa(A)).
\end{tikzcd} \tag{3.2.1}$$

Hence

$$H^G(X, \kappa(A)) = \mathrm{hocofiber}(H^G(X, KH(\mathcal{L}^1 \otimes_{\mathbb{C}} A))[+1] \to H^G(X, HC(A/\mathbb{C}))[-1]). \tag{3.2.2}$$

Similarly, a choice of rank one projection induces a map of OrG-spectra

$$K(A[\mathcal{G}(G/H)]) \to K(\mathcal{B} \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)] : \mathcal{L}^1 \otimes_{\mathbb{C}} A[\mathcal{G}(G/H)]).$$

Taking equivariant homology we obtain a map

$$H^G(X, K(A)) \to H^G(X, K(\mathcal{B} \otimes_{\mathbb{C}} A : \mathcal{L}^1 \otimes_{\mathbb{C}} A)).$$

Composing this map with the bottom arrow in diagram (3.2.1) we obtain an equivariant character

$$\tau : H^G(X, K(A)) \to H^G(X, \kappa(A)). \tag{3.2.3}$$

In what follows we shall be interested in several families of finite subgroups of a given group. As usual, we write $\mathcal{F}in$ for the family of finite subgroups, and $\mathcal{F}cyc$ for the subfamily of those finite subgroups that are cyclic. If $m \geq 1$ we write $\mathcal{C}yc_m$ for the family of those cyclic subgroups whose order divides m. If $H \subset G$ is a subgroup in the family $\mathcal{F}$, we write (H) for the conjugacy class of H and

$$(\mathcal{F}) = \{(H) : H \in \mathcal{F}\}$$

for the set of all conjugacy classes of subgroups of G in the family $\mathcal{F}$.

For $C \subset G$ a cyclic subgroup let $N_G C = \{g \in G : gCg^{-1} = C\}$ be its normalizer, let $Z_G C = \{g \in G : gcg^{-1} = c \ \forall c \in C\}$ be its centralizer, and put

$$W_G C = N_G C / Z_G C.$$

Let $C \subset G$ be a finite cyclic group and $A(C)$ its Burnside ring. Recall that $A(C)$ is the ring of formal differences of isomorphism classes of finite C-sets, with addition given by disjoint union and multiplication given by cartesian product. The abelian groups $K_*^{\mathrm{top}}(\mathbb{C}[C])$ and $HC_*(\mathbb{C}[C])$ become modules over $A(C)$. There is a ring homomorphism

$$A(C) \to \prod_{(H)\in\mathcal{F}inC} \mathbb{Z}, \quad [S] \to (|S^H|)_{(H)\in\mathcal{F}inC}$$

which sends the class of a finite C-set to the element given by the cardinality of the H-fixed point sets. This is an injection with finite cokernel. Hence, leads to an isomorphism

$$A(C) \otimes \mathbb{Q} \overset{\cong}{\to} \prod_{(H)\in\mathcal{F}inC} \mathbb{Q}.$$

We write $\theta_C \in A(C) \otimes \mathbb{Q}$ for the element corresponding to the characteristic function $\chi_C \in \prod_{(H)\in\mathcal{F}inC} \mathbb{Q}$.

Proposition 3.2.4. *Let G be a group. Then the map*

$$H_*^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \to H_*^G(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C}))$$

is an isomorphism and

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q} = \bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{F}cyc)} H_p(Z_G C, \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot \kappa_q(\mathbb{C}[C]) \otimes \mathbb{Q}. \quad (3.2.5)$$

Proof. If H is a finite subgroup, then the equivalence $KH(\mathcal{L}^1) \xrightarrow{\sim} K^{\mathrm{top}}(\mathcal{L}^1) \overset{\sim}{\leftarrow} K^{\mathrm{top}}(\mathbb{C})$ induces an equivalence $KH(\mathcal{L}^1[\mathcal{G}(G/H)]) \overset{\sim}{\to} K^{\mathrm{top}}(C^*(\mathcal{G}(G/H)))$. Hence if X is a $(G, \mathcal{F}in)$ -complex, we have an equivalence

$$H^G(X, KH(\mathcal{L}^1)) \xrightarrow{\sim} H^G(X, K^{\mathrm{top}}(\mathbb{C})). \quad (3.2.6)$$

Thus $H^G(\mathcal{E}(G, \mathcal{F}cyc), KH(\mathcal{L}^1)) \to H^G(\mathcal{E}(G, \mathcal{F}in), KH(\mathcal{L}^1))$ is an equivalence because $H^G(\mathcal{E}(G, \mathcal{F}cyc), K^{\mathrm{top}}(\mathbb{C})) \to H^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathbb{C}))$ is ([26, Proposition 69]). Similarly, $H^G(\mathcal{E}(G, \mathcal{F}cyc), HC(\mathbb{C}/\mathbb{C})) \to H^G(\mathcal{E}(G, \mathcal{F}in), HC(\mathbb{C}/\mathbb{C}))$ is an equivalence (see [27, §9] or [6, §7]). From (3.2.2) and what we have just proved, it follows that

$$H^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \to H^G(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C}))$$

is an equivalence. This shows the first assertion of the proposition. From (3.2.6), [25, Theorem 0.7] and [26, Theorem 172], we get

$$H_n^G(\mathcal{E}(G,\mathcal{F}cyc),KH(\mathcal{L}^1))\otimes\mathbb{Q}=\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C\cdot K_q^{\mathrm{top}}(\mathbb{C}[C])\otimes\mathbb{Q}.$$

Next write $\mathrm{con}_f(G)$ for the conjugacy classes of elements of G of finite order, and $Gen(C)$ for the set of all generators of $C\in\mathcal{F}cyc$. By using [27, Lemma 7.4] and the argument of the proof of [7, Proposition 2.2.1] we obtain

$$\begin{aligned}H_n^G(\mathcal{E}(G,\mathcal{F}cyc),HC(\mathbb{C}/\mathbb{C}))&=\bigoplus_{p+q=n}\bigoplus_{(g)\in\mathrm{con}_f(G)}H_p(Z_G\langle g\rangle,\mathbb{Q})\otimes HC_q(\mathbb{C}/\mathbb{C})\\&=\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_G(C)]}\mathrm{map}(Gen(C),HC_q(\mathbb{C}/\mathbb{C}))\\&=\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_G(C)]}\theta_C\cdot HC_q(\mathbb{C}[C]/\mathbb{C}).\end{aligned}$$

It follows from the proof of Proposition 3.1.2.2 that under the isomorphism (3.2.6) and the identity $H_*^G(-,HC(\mathbb{C}/\mathbb{C}))=H_*^G(-,HC^{\mathrm{top}}(\mathbb{C}))$ the map $\mathrm{Tr}Sch$ identifies with $\mathrm{Tr}Sch^{\mathrm{top}}$. Hence, by naturality, the map

$$(\mathrm{Tr}Sch)_n:H_{n+1}^G(\mathcal{E}(G,\mathcal{F}cyc),KH(\mathcal{L}^1))\to H_{n-1}^G(\mathcal{E}(G,\mathcal{F}cyc),HC(\mathbb{C}/\mathbb{C}))$$

is induced by the maps $\mathrm{Tr}Sch_q^{\mathrm{top}}:K_{q+1}^{\mathrm{top}}(\mathbb{C}[C])\to HC_{q-1}^{\mathrm{top}}(\mathbb{C}[C])=HC_{q-1}(\mathbb{C}[C]/\mathbb{C})$. The computation of $H_n^G(\mathcal{E}(G,\mathcal{F}in),\kappa(\mathbb{C}))\otimes\mathbb{Q}$ is now immediate from this. □

Remark 3.2.7. We have an equivalence of $(G,\mathcal{F}cyc)$-spaces

$$\operatorname*{colim}_m\mathcal{E}(G,\mathcal{C}yc_m)\xrightarrow{\cong}\mathcal{E}(G,\mathcal{F}cyc)$$

where the colimit is taken with respect to the partial order of divisibility. Hence for every $\mathrm{Or}G$-spectrum E,

$$H_*^G(\mathcal{E}(G,\mathcal{F}cyc),E)=\operatorname*{colim}_m H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),E).$$

Moreover it is clear from the proof of Proposition 3.2.4 that for every m the map

$$H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C}))\otimes\mathbb{Q}\to H_*^G(\mathcal{E}(G,\mathcal{F}cyc),\kappa(\mathbb{C}))\otimes\mathbb{Q}$$

is the inclusion

$$\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{C}yc_m)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C\cdot\kappa_q(\mathbb{C}[C])\otimes\mathbb{Q}\hookrightarrow$$
$$\bigoplus_{p+q=n}\bigoplus_{(C)\in(\mathcal{F}cyc)}H_p(Z_GC,\mathbb{Q})\otimes_{\mathbb{Q}[W_GC]}\theta_C\cdot\kappa_q(\mathbb{C}[C])\otimes\mathbb{Q}.$$

3.3 Conditions equivalent to the rational injectivity of the κ assembly map

Let G be a group. As shown in the proof of Proposition (3.2.4), we have a direct sum decomposition

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), HC(\mathbb{C}/\mathbb{C})) = \bigoplus_{p+q=n} \bigoplus_{(g)\in \mathrm{con}_f(G)} H_p(Z_G\langle g\rangle, \mathbb{Q}) \otimes HC_q(\mathbb{C}/\mathbb{C}).$$

On the other hand we also have a decomposition

$$HC_n(\mathbb{C}[G]/\mathbb{C}) = \bigoplus_{(g)\in \mathrm{con}(G)} HC_n^{(g)}(\mathbb{C}[G]/\mathbb{C}).$$

The assembly map identifies

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), HC(\mathbb{C}/\mathbb{C})) = \bigoplus_{(g)\in \mathrm{con}_f(G)} HC_n^{(g)}(\mathbb{C}[G]/\mathbb{C}).$$

Thus there is a projection

$$\pi_n^{\mathcal{F}cyc} : HC_n(\mathbb{C}[G]/\mathbb{C}) \to H_n^G(\mathcal{E}(G, \mathcal{F}cyc), HC(\mathbb{C}/\mathbb{C}))$$

which is left inverse to the asembly map.
By composing $KH_{n+1}(\mathcal{L}^1[G]) \overset{\mathrm{Tr}Sch}{\to} HC_{n-1}(\mathbb{C}[G]/\mathbb{C})$ with the projection above, we obtain a map

$$\pi_{n-1}^{\mathcal{F}cyc}\mathrm{Tr}Sch : KH_{n+1}(\mathcal{L}^1[G]) \otimes \mathbb{Q} \to H_{n-1}^G(\mathcal{E}(G, \mathcal{F}cyc), HC(\mathbb{C}/\mathbb{C})). \tag{3.3.1}$$

Next, if $m \geq 1$ then

$$H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), HC(\mathbb{C}/\mathbb{C})) = \bigoplus_{(g)\in \mathrm{con}_f(G), g^m=1} HC_n^{(g)}(\mathbb{C}[G]/\mathbb{C}).$$

Thus we also have a map

$$\pi_{n-1}^{\mathcal{C}yc_m}\mathrm{Tr}Sch : KH_{n+1}(\mathcal{L}^1[G]) \otimes \mathbb{Q} \to H_{n-1}^G(\mathcal{E}(G, \mathcal{C}yc_m), HC(\mathbb{C}/\mathbb{C})). \tag{3.3.2}$$

In the following proposition we use the following notation. We write

$$H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), \kappa(\mathbb{C}) \otimes \mathbb{Q})^+ := \bigoplus_{p+q=n, q\geq 1} \bigoplus_{(C)\in(\mathcal{C}yc_m)} H_p(Z_G C, \mathbb{Q}) \otimes_{\mathbb{Q}[W_G C]} \theta_C \cdot \kappa_q(\mathbb{C}[C]) \otimes \mathbb{Q} \tag{3.3.3}$$

and

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C})\otimes\mathbb{Q})^- := \bigoplus_{p+q=n, q\le 0}\ \bigoplus_{(C)\in(\mathcal{C}yc_m)} H_p(Z_G C,\mathbb{Q})\otimes_{\mathbb{Q}[W_G C]}\theta_C\cdot\kappa_q(\mathbb{C}[C])\otimes\mathbb{Q}. \quad (3.3.4)$$

Note that, by Proposition 3.2.4, $H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C}))\otimes\mathbb{Q}$ is the direct sum of (3.3.3) and (3.3.4).

Proposition 3.3.5. *Let G be a group, $n\in\mathbb{Z}$ and $m\ge 1$. The following are equivalent.*

i) The rational assembly map

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C}))\otimes\mathbb{Q}\to\kappa_n(\mathbb{C}[G])\otimes\mathbb{Q} \quad (3.3.6)$$

is injective.

ii) The restriction of the rational assembly map to the summand (3.3.3) *is injective.*

iii) The image of the map (3.3.2) *coincides with the image of*

$$\mathrm{Tr}Sch : H_{n+1}^G(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q}\to H_{n-1}^G(\mathcal{E}(G,\mathcal{C}yc_m),HC(\mathbb{C}/\mathbb{C})).$$

Proof. It is clear that i) implies ii). Assume that ii) holds and consider the following commutative diagram with exact columns:

$$\begin{tikzcd}
H_{n+1}^G(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q} \arrow[r] \arrow[d, "\mathrm{Tr}Sch"] & KH_{n+1}(\mathcal{L}^1[G])\otimes\mathbb{Q} \arrow[d] \arrow[dl, "(3.3.2)"] \\
H_{n-1}^G(\mathcal{E}(G,\mathcal{C}yc_m),HC(\mathbb{C}/\mathbb{C})) \arrow[r] \arrow[d] & HC_{n-1}(\mathbb{C}[G]/\mathbb{C}) \arrow[d] \\
H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C}))\otimes\mathbb{Q} \arrow[r] \arrow[d] & \kappa_n(\mathbb{C}[G])\otimes\mathbb{Q} \arrow[d] \\
H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q} \arrow[r] & KH_n(\mathcal{L}^1[G])\otimes\mathbb{Q}.
\end{tikzcd}$$

Let x be an element of the kernel of the map of part i), that is of the first map above bottom in the diagram above. Write $x=x_++x_-$, with x_+ in (3.3.3) and x_- in (3.3.4). The image of x under the vertical map must be zero, since by Yu's theorem ([36], see also [7]), the bottom horizontal map is injective. By (3.1.1.3) and the proof of Proposition 3.2.4, this implies that $x_-=0$, proving that ii) implies i). Next assume y is an element in the image of (3.3.2) which is not in the image of the vertical map $\mathrm{Tr}Sch$ in the diagram above. Then the image of y under the vertical map is a nonzero element of the kernel of the next horizontal map. Thus i) implies iii). The converse is also clear, using Yu's theorem again. $\square$

Corollary 3.3.7. *Let G be a group and let $n,p\in\mathbb{Z}$ with $n\equiv p+1 \mod 2$. Assume that the map*

$$H_p^G(\mathcal{E}(G,\mathcal{F}cyc),KH(\mathcal{L}^1))\otimes\mathbb{Q}\to KH_p(\mathcal{L}^1[G])\otimes\mathbb{Q} \quad (3.3.8)$$

is surjective. Then the map (3.3.6) *is injective for every* $m \geq 1$.

Proof. By Yu's theorem ([36],[7]) the map (3.3.8) is always injective; under our current assumptions, it is an isomorphism. Moreover, by [10, Theorem 6.5.3], the groups $KH_p(\mathcal{L}^1[G])$ depend only on the parity of p. It follows that condition iii) of Proposition 3.3.5 holds for every m and every $n \equiv p+1 \mod 2$. This concludes the proof. □

3.4 Rational injectivity of the equivariant regulators

Let G be a group. By composing the equivariant character (3.2.3) with the map induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$ we obtain a map

$$H_*^G(\mathcal{E}(G,\{1\}), K(\mathbb{Z})) \to H_*^G(\mathcal{E}(G,\mathcal{F}in), K(\mathbb{C})) \xrightarrow{\tau} H_*^G(\mathcal{E}(G,\mathcal{F}in), \kappa(\mathbb{C})). \tag{3.4.1}$$

Proposition 3.4.2. *The map* (3.4.1) *is rationally injective.*

Proof. By [26, Remark 12] we have

$$H_n^G(\mathcal{E}(G,\{1\}), K(\mathbb{Z})) \otimes \mathbb{Q} = \bigoplus_{p+q=n} H_p(G,\mathbb{Q}) \otimes_{\mathbb{Q}} K_q(\mathbb{Z}) \otimes \mathbb{Q}. \tag{3.4.3}$$

By Theorem 3.1.3.1, the regulators $K_q(\mathbb{Z}) \to K_q(\mathbb{C}) \to \kappa_q(\mathbb{C}) = MK_q(\mathbb{C})$ induce a monomorphism from (3.4.3) to

$$\bigoplus_{p+q=n} H_p(G,\mathbb{Q}) \otimes_{\mathbb{Q}} \kappa_q(\mathbb{C}) \otimes \mathbb{Q}. \tag{3.4.4}$$

The map (3.4.1) tensored with $\mathbb{Q}$ is the composite of the above monomorphism with the inclusion of (3.4.4) as a direct summand in (3.2.5). □

Let F be a number field, G a group and $m \geq 1$. Let ζ_m be a primitive m^{th} root of 1. The map $\mathcal{E}(G, \mathcal{C}yc_m) \to \mathcal{E}(G, \mathcal{F}cyc)$, together with the inclusion

$$F \subset F(\zeta_m) \xrightarrow{\iota} \mathbb{C}^{\hom(F(\zeta_m),\mathbb{C})}$$

and the character $\tau : K(\mathbb{C}) \to \kappa(\mathbb{C})$, induce a homomorphism

$$H_*^G(\mathcal{E}(G,\mathcal{C}yc_m), K(F)) \to H_*^G(\mathcal{E}(G,\mathcal{F}cyc), \kappa(\mathbb{C}))^{\hom(F(\zeta_m),\mathbb{C})}. \tag{3.4.5}$$

Proposition 3.4.6. *The map* (3.4.5) *is rationally injective.*

Proof. By [24, Theorem 0.3], we have

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m), K(F)) = \bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{C}yc_m)} H_p(Z_GC,\mathbb{Q}) \otimes_{\mathbb{Q}[Z_GC]} \theta_C \cdot K_q(F[C]) \otimes \mathbb{Q}. \tag{3.4.7}$$

By Lemma 3.1.3.4 the maps $K_q(F[C]) \to \kappa_q(\mathbb{C}[C])^{\hom(F(\zeta_m),\mathbb{C})}$ with $C \in \mathcal{C}yc_m$ induce a rational monomorphism from (3.4.7) to

$$\bigoplus_{p+q=n} \bigoplus_{(C)\in(\mathcal{C}yc_m)} H_p(Z_GC,\mathbb{Q}) \otimes_{\mathbb{Q}[Z_GC]} \theta_C \cdot \kappa_q(\mathbb{C}[C])^{\hom(F(\zeta_m),\mathbb{C})}. \tag{3.4.8}$$

The map (3.4.5) tensored with $\mathbb{Q}$ is the composite of the above monomorphism with the inclusion of (3.4.8) as a summand in (3.2.5). □

3.5 Comparing conjectures and assembly maps

Theorem 3.5.1. *Let G be a group. Assume that the equivalent conditions of Proposition 3.3.5 hold for G with $m = 1$. Then the assembly map*

$$H_n^G(\mathcal{E}(G,\{1\}), K(\mathbb{Z})) \to K_n(\mathbb{Z}[G])$$

is rationally injective. In particular this is the case whenever G satisfies the rational KH-isomorphism conjecture with $\mathcal{L}^1$-coefficients.

Proof. Immediate from Proposition 3.4.2 and Corollary 3.3.7. □

Theorem 3.5.2. *Let G be a group and $m \geq 1$. Assume that the equivalent conditions of Proposition 3.3.5 hold for G and m. Then for every number field F, the assembly map*

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m), K(F)) \to K_n(F[G])$$

is rationally injective. If moreover the condition holds for all m –as is the case, for example, if G satisfies the rational KH-isomorphism conjecture with $\mathcal{L}^1$-coefficients- then G satisfies the rational injectivity part of the K-theory isomorphism conjecture with coefficients in any number field.

Proof. Immediate from Proposition 3.4.6 and Corollary 3.3.7. □

3.6 Resumen

Sean G un grupo, $\mathcal{F}in$ la familia de sus subgrupos finitos, y $\mathcal{L}^1$ el álgebra de operadores de traza en un espacio de Hilbert de dimensión infinita, complejo y separable. En el capítulo anterior probamos que el morfismo de ensamble racional para la K-teoría homotópica

$$H_p^G(\mathcal{E}(G,\mathcal{F}in), KH(\mathcal{L}^1)) \otimes \mathbb{Q} \to KH_p(\mathcal{L}^1[G]) \otimes \mathbb{Q} \tag{3.6.1}$$

es inyectivo. La conjetura de isomorfismo racional para KH ([2, Conjecture 7.3]) predice que (3.6.1) es un isomorfismo. En este capítulo probamos los siguiente

Teorema 3.6.2. *Supongamos que* (3.6.1) *es suryectivo. Sea* $n \equiv p+1 \mod 2$. *Entonces:*
i) El morfismo de ensamble racional para la familia trivial

$$H_n^G(\mathcal{E}(G,\{1\}), K(\mathbb{Z})) \otimes \mathbb{Q} \to K_n(\mathbb{Z}[G]) \otimes \mathbb{Q} \tag{3.6.3}$$

es inyectivo.
ii) Para todo cuerpo de números F, *el morfismo de ensamble racional*

$$H_n^G(\mathcal{E}(G,\mathcal{F}in), K(F)) \otimes \mathbb{Q} \to K_n(F[G]) \otimes \mathbb{Q} \tag{3.6.4}$$

es inyectivo.

Observemos que la conjetura de Novikov para K-teoría afirma que el inciso i) del teorema anterior vale para todo G, y que el inciso ii) es equivalente a la inyectividad racional de la conjetura de Farrell-Jones para K-teoría con coeficientes en un cuerpo de números ([26, Conjeturas 51 y 58 y Proposción 70]).

La idea de la demostración del Teorema 3.6.2 es usar una versión algebraica y equivariante de la K-teoría multiplicativa de Karoubi. Esta teoría asigna grupos $MK_n(\mathfrak{A})$ $(n \geq 1)$ para cada álgebra de Banach unital $\mathfrak{A}$, que forman parte de una sucesión exacta larga

$$HC_{n-1}^{\mathrm{top}}(\mathfrak{A}) \to MK_n(\mathfrak{A}) \to K_n^{\mathrm{top}}(\mathfrak{A}) \overset{Sch_n^{\mathrm{top}}}{\to} HC_{n-2}^{\mathrm{top}}(\mathfrak{A}).$$

Aquí HC^{top} es la homología cíclica del módulo cíclico completo $C_n^{\mathrm{top}}(\mathfrak{A}) = \mathfrak{A} \otimes_\pi \ldots \otimes_\pi \mathfrak{A}$ ($n+1$ factores), ch^{top} es el caracter de Chern de Connes-Karoubi con valores en la homología cíclica periódica $HP_*^{\mathrm{top}}(\mathfrak{A})$, y S es el operador de periodicidad. Karoubi introdujo un caracter de Chern multiplicativo

$$\mu_n : K_n(\mathfrak{A}) \to MK_n(\mathfrak{A}). \tag{3.6.5}$$

En particular si $\mathcal{O}$ es el anillo de enteros en un cuerpo de números F se puede considerar la composición

$$K_n(\mathcal{O}) \to K_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^{\hom(F,\mathbb{C})}. \tag{3.6.6}$$

Comparando esta aplicación con el regulador de Borel, Karoubi mostró en [22] que (3.6.6) es racionalmente inyectiva. De esto se deduce que

$$K_n(\mathbb{Z}) \to MK_n(\mathbb{C}) \tag{3.6.7}$$

es racionalmente inyectiva. En este capítulo asignamos, a cada $\mathbb{C}$-álgebra unital A, grupos $\kappa_n(A)$ $(n \in \mathbb{Z})$ que forman parte de una sucesión exacta larga

$$HC_{n-1}(A/\mathbb{C}) \to \kappa_n(A) \to KH_n(\mathcal{L}^1 \otimes_{\mathbb{C}} A) \overset{\mathrm{Tr}Sch_n}{\to} HC_{n-2}(A/\mathbb{C}).$$

Aquí $HC(/\mathbb{C})$ es la homología cíclica de $\mathbb{C}$-álgebras, ch es el caracter de Chern algebraico de Connes-Karoubi y Tr es la aplicación inducida por el operador de traza. Introducimos además un caracter

$$\tau_n : K_n(A) \to \kappa_n(A) \quad (n \in \mathbb{Z}). \tag{3.6.8}$$

Si $\mathfrak{A}$ es un álgebra de Banach de dimensión finita y $n \geq 1$ luego $\kappa_n(\mathfrak{A}) = MK_n(\mathfrak{A})$ y (3.6.5) se identifica con (3.6.8) (Proposición 3.1.2.2). Tanto κ como τ tienen versiones equivariantes, por lo tanto si X es un G-espacio y A es una $\mathbb{C}$-álgebra, obtenemos un morfismo de ensamble

$$H_n^G(X, \kappa(A)) \to \kappa_n(A[G]).$$

Sea $\mathcal{F}cyc$ la familia de subgrupos cíclicos finitos. En la Proposición 3.2.4 mostramos que el morfismo

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \to H_n^G(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C}))$$

es un isomorfismo, y calculamos $H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q}$ en términos de los subgrupos finitos cíclicos de G. Usamos esto y la inyectividad racional de (3.6.7) para probar, en la Proposición 3.4.2, que el morfismo

$$H_n^G(\mathcal{E}(G, \{1\}), K(\mathbb{Z})) \to H_n^G(\mathcal{E}(G, \mathcal{F}in), K(\mathbb{C})) \xrightarrow{\tau} H_n^G(\mathcal{E}(G, \mathcal{F}in), \kappa(\mathbb{C})) \tag{3.6.9}$$

es racionalmente inyectivo. Por [26, Proposición 76] sabemos que el morfismo

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), K(R)) \otimes \mathbb{Q} \to H_n^G(\mathcal{E}(G, \mathcal{F}in), K(R)) \otimes \mathbb{Q}$$

es un isomorfismo para todo anillo unitario R. En particular, podemos sustituir $\mathcal{F}cyc$ por $\mathcal{F}in$ en (3.6.4). Usamos esto junto con la Proposición 3.2.4 y la inyectividad racional de

$$K_n(F) \to K_n(\mathbb{C})^{\hom(F,\mathbb{C})} \to MK_n(\mathbb{C})^{\hom(F,\mathbb{C})}$$

(ver Observación 3.1.3.2), para mostrar en la Proposición 3.4.6 que si $m \geq 1$, $\mathcal{C}yc_m$ es la familia de subgrupos cíclicos cuyo orden divide a m, y ζ_m es una raíz primitiva de la unidad de orden m, entonces la composición

$$\begin{tikzcd} H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), K(F)) \otimes \mathbb{Q} \arrow[r] \arrow[dr, dotted] & H_n^G(\mathcal{E}(G, \mathcal{C}yc_m), K(\mathbb{C}))^{\hom(F(\zeta_m),\mathbb{C})} \otimes \mathbb{Q} \arrow[d, "\tau"] \\ & H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C}))^{\hom(F(\zeta_m),\mathbb{C})} \otimes \mathbb{Q} \end{tikzcd} \tag{3.6.10}$$

es inyectiva. Dado que el morfismo $\operatorname{colim}_m \mathcal{E}(G, \mathcal{C}yc_m) \to \mathcal{E}(G, \mathcal{F}cyc)$ es una equivalencia, se sigue que si el morfismo de ensamble racional

$$H_n^G(\mathcal{E}(G, \mathcal{F}cyc), \kappa(\mathbb{C})) \otimes \mathbb{Q} \to \kappa_n(\mathbb{C}[G]) \otimes \mathbb{Q} \tag{3.6.11}$$

es inyectivo también los son (3.6.3) y (3.6.4). En el Corolario 3.3.7 mostramos que si (3.6.1) es suryectivo, entonces (3.6.11) es inyectivo para $n \equiv p + 1 \mod 2$. Esto prueba el Teorema 3.6.2.

El resto de este capítulo está organizado de la siguiente manera. En la Sección 3.1 definimos $\kappa_n(A)$ y el morfismo $\tau_n : K_n(A) \to \kappa_n(A)$. Por definición, si $n \leq 0$ $\kappa_n(A) = KH_n(A \otimes_{\mathbb{C}} \mathcal{L}^1)$ y τ_n es el morfismo identidad (3.1.1.3). En la Proposición 3.1.2.2 mostramos que si $n \geq 1$ y $\mathfrak{A}$ es un álgebra de Banach de dimensión finita, entonces $\kappa_n(\mathfrak{A}) = MK_n(\mathfrak{A})$ y $\tau_n = \mu_n$. En el Teorema 3.1.3.1 recordamos los reguladores de Karoubi y sus resultados sobre inyectividad. Usamos el teorema de Karoubi para probar, en el Lema 3.1.3.4, que si F es un cuerpo de números, C es un grupo cíclico de orden m, n es un múltiplo de m, y ζ_n es una raíz primitiva de la unidad de orden n, luego la composición

$$K_*(F[C]) \to K_*(F(\zeta_n)[C]) \to K_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})} \xrightarrow{\mu} MK_*(\mathbb{C}[C])^{\hom(F(\zeta_n),\mathbb{C})}$$

es racionalmente inyectiva. El resultado principal de la Sección 3.2 es la Proposición 3.2.4, que calcula $H_n^G(\mathcal{E}(G,\mathcal{F}cyc),\kappa(\mathbb{C}))\otimes\mathbb{Q}$ en términos de homología de grupos y de los grupos $\kappa_*(\mathbb{C}[C])$ para $C\in\mathcal{F}cyc$. La fórmula resultante es similar a las ya existentes para K-homología y homología cíclica equivariantes, que se usan en la prueba ([6], [24], [25], [26],[27]). En la Sección 3.3 mostramos que si el morfismo de ensamble racional para $KH(\mathcal{L}^1)$ es suryectivo, entonces el morfismo de ensamble racional para $\kappa(\mathbb{C})$ resulta inyectivo (Corolario 3.3.7). Para esto utilizamos el hecho de que para todo $m\geq 1$, el morfismo de ensamble

$$H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),HC(\mathbb{C}/\mathbb{C}))\to HC_*(\mathbb{C}[G])$$

tiene una inversa a izquierda natural π_m, que hace conmutar el siguiente diagrama

$$\begin{tikzcd}
H_*^G(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q} \arrow[r] \arrow[d, "\mathrm{Tr}Sch"] & KH_*(\mathcal{L}^1[G])\otimes\mathbb{Q} \arrow[d, "\mathrm{Tr}Sch"] \\
H_{*-2}^G(\mathcal{E}(G,\mathcal{C}yc_m),HC(\mathbb{C}/\mathbb{C})) & HC_{*-2}(\mathbb{C}[G]). \arrow[l, "\pi_m"']
\end{tikzcd}$$

Luego para cada n se tiene una inclusión

$$\mathrm{Tr}Sch(H_{n+1}^G(\mathcal{E}(G,\mathcal{C}yc_m),KH(\mathcal{L}^1))\otimes\mathbb{Q})\subset\pi_m\mathrm{Tr}Sch(KH_{n+1}(\mathcal{L}^1[G])\otimes\mathbb{Q}). \tag{3.6.12}$$

En la Proposición 3.3.5 mostramos que el morfismo de ensamble racional

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),\kappa(\mathbb{C}))\otimes\mathbb{Q}\to\kappa_n(\mathbb{C}[G])\otimes\mathbb{Q}$$

es inyectivo si y sólo si la inclusión (3.6.12) es una igualdad. De esto se deduce inmediatamente el Corolario 3.3.7. En la Sección 3.4 probamos la inyectividad de (3.6.9) y (3.6.10) (Proposiciones 3.4.2 y 3.4.6). Finalmente en la Sección 3.5 mostramos que si en (3.6.12) vale la igualdad para $m=1$ luego (3.6.3) es inyectivo (Teorema 3.5.1) y que si vale para m, luego

$$H_n^G(\mathcal{E}(G,\mathcal{C}yc_m),K(F))\otimes\mathbb{Q}\to K_n(F[G])\otimes\mathbb{Q}$$

es inyectivo para todo cuerpo de números F (Teorema 3.5.2).

Estos resultados fueron publicados en [8].

4. COMPACT OPERATORS AND ALGEBRAIC K-THEORY FOR GROUPS WHICH ACT PROPERLY AND ISOMETRICALLY ON HILBERT SPACE

In this chapter we are interested in the K-theory isomorphism conjecture for coefficient rings of the form

$$R = I \otimes (\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K}) \tag{4.1}$$

where I is a K-excisive G-ring, $\mathfrak{A}$ is a complex G-C^*-algebra (or more generally a bornolocal C^*-algebra as defined in Section 4.1), $\underset{\sim}{\otimes}$ is the spatial tensor product (see Appendix A.2.1), and $\mathcal{K}$ is the ideal of compact operators in an infinite dimensional, separable, complex Hilbert space with trivial G-action. We consider the Farrell-Jones conjecture for discrete groups having the *Haagerup approximation property.* These are the countable discrete groups which admit an affine, isometric and *metrically proper* action on a real pre-Hilbert space V of countably infinite dimension (or equivalently on a Hilbert space). The term metrically proper means that for every $v \in V$,

$$\lim_{g\to\infty} ||gv|| = \infty.$$

The groups satisfying this property are also called *a-T-menable*, a term coined by Gromov ([15]). Our main result is the following (see Theorem 4.6.1).

Theorem 4.2. *Let G be a countable discrete group. Let $\mathfrak{A}$ be a G-C^*-algebra, let $I \in G$-Ring, and let $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ be the algebra of compact operators; equip $\mathcal{K}$ with the trivial G-action. Assume that I is K-excisive and that G has the Haagerup approximation property. Then the functor $H^G(-, K(I \otimes (\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K})))$ sends $\mathcal{F}in$-equivalences of G-spaces to weak equivalences of spectra.*

Observe that because $\mathcal{V}cyc \supset \mathcal{F}in$, any $\mathcal{V}cyc$-equivalence is also a $\mathcal{F}in$-equivalence. Since $\mathcal{E}(G, \mathcal{V}cyc) \to pt$ is a $\mathcal{V}cyc$-equivalence by definition, the theorem has the following corollary (see Corollary 4.6.3).

Corollary 4.3. *Let G, I and $\mathfrak{A}$ be as in Theorem 4.2. Then G satisfies the K-theoretic Farrell-Jones conjecture with coefficients in $I \otimes (\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K})$.*

Higson and Kasparov proved in [20] that the groups which have the Haagerup approximation property satisfy the *Baum-Connes conjecture* with coefficients in any separable G-C^*-algebra. Recall that the latter conjecture is the analogue of the Farrell-Jones conjecture for the topological K-theory of reduced C^*-crossed products (for the definitions of crossed products see Appendix C). It asserts that the assembly map

$$H^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathfrak{A})) \to K^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A}))$$

is a weak equivalence.

There is a natural map

$$\mathfrak{A} \rtimes H \to C^*_{\mathrm{red}}(H, \mathfrak{A}) \tag{4.4}$$

which is an isomorphism when H is finite. We have a homotopy commutative diagram

$$\begin{tikzcd} H^G(\mathcal{E}(G, \mathcal{F}in), K(\mathfrak{A})) \arrow[r] \arrow[d] & K(\mathfrak{A} \rtimes G) \arrow[d] \\ H^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathfrak{A})) \arrow[r] & K^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A})). \end{tikzcd} \tag{4.5}$$

It follows from Suslin-Wodzicki's theorem (Karoubi's conjecture) ([31, Theorem 10.9]) and the facts that (4.4) is an isomorphism for finite H, and that G acts on $\mathcal{E}(G, \mathcal{F}in)$ with finite stabilizers, that the vertical map on the left of (4.5) is a weak equivalence whenever $\mathfrak{A}$ is of the form $\mathfrak{A} = \mathfrak{B}\underset{\sim}{\otimes}\mathcal{K}$. Using this, the stability of K^{top} under tensoring with $\mathcal{K}$, and Higson-Kasparov's result, we obtain the following corollary of Theorem 4.2 (see Corollary 4.6.5).

Corollary 4.6. *Let G and $\mathfrak{A}$ be as in Theorem 4.2. Assume that $\mathfrak{A}$ is separable. Then there is an isomorphism:*

$$K_*((\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K}) \rtimes G) \cong K^{\mathrm{top}}_*(C^*_{\mathrm{red}}(G, \mathfrak{A})).$$

Higson and Kasparov showed in [20, Theorem 7.1] that if G is a locally compact group which has the Haagerup property, $\mathfrak{A}$ is a separable G-C^*-algebra, and $C^*(G, \mathfrak{A})$ is the full crossed product, then the map

$$K^{\mathrm{top}}_*(C^*(G, \mathfrak{A})) \to K^{\mathrm{top}}_*(C^*_{\mathrm{red}}(G, \mathfrak{A}))$$

is an isomorphism. Hence in Corollary 4.6 we may substitute the full C^*-crossed product for the reduced one.

The rest of this chapter is organized as follows. In Section 4.1 we give some preliminaries on bornolocal C^*-algebras. These are normed $*$-algebras over $\mathbb{C}$ such that $\|a^*a\| = \|a\|^2$, possibly not complete, which are filtered unions of C^*-subalgebras. For example, if X is a locally compact Hausdorff topological space and $\mathfrak{A}$ is a C^*-algebra, then the algebra $C_c(X, \mathfrak{A})$ of compactly supported continuous functions $X \to \mathfrak{A}$ is a bornolocal C^*-algebra. We write $\mathfrak{BC}^*$ for the category of bornolocal C^*-algebras and G-$\mathfrak{BC}^*$ for the corresponding equivariant category. In Section 4.2 we prove Theorem 4.2.2.4, which says that if G is a discrete group, X a G-space, I a K-excisive G-ring and $B \in G$-$\mathfrak{BC}^*$, then the functor

$$\mathbb{E}_X : G\text{-}\mathfrak{BC}^* \to \mathrm{Spt}, \quad A \mapsto H^G(X, K(I \otimes A \otimes_\mu \mathcal{K} \otimes_\mu B)) \tag{4.7}$$

is excisive, homotopy invariant, and G-stable. Here $\otimes_\mu$ is the maximal tensor product of bornolocal C^*-algebras; it is defined in (4.2.2.1). Section 4.3 concerns equivariant asymptotic homomorphisms of G-bornolocal C^*-algebras. In this technical section, we discuss how to extend the functor (4.7) to a functor $\bar{\mathbb{E}}_X$ that can be applied to certain equivariant asymptotic homomorphisms; the main results of this section are Proposition 4.3.3.7 and Lemma 4.3.3.13. In Section 4.4 we recall Higson-Kasparov's construction of a dual Dirac element in equivariant E-theory ([20]). For a group G which acts by affine isometries on a countably infinite dimensional Euclidean space V, they construct a G-C^*-algebra $\mathcal{A}_0(V)$ which is a C^*-colimit over all finite dimensional subspaces $S \subset V$, of algebras of continuous functions $\mathbb{R} \times S \to \mathrm{Cliff}(\mathbb{R} \oplus S)$, vanishing at infinity, and taking values in the complexified Clifford algebra $\mathrm{Cliff}(\mathbb{R} \oplus S)$. They define an equivariant asymptotic homomorphism

$$\hat{\beta}_0 : C_0(\mathbb{R}) - - > \mathcal{A}_0(V), \tag{4.8}$$

and they show that its class in $E_G(C_0(\mathbb{R}), \mathcal{A}_0(V))$, which they call the *dual Dirac element*, is invertible. We define a bornolocal G-C^*-algebra $\mathcal{A}_c(V)$ which is an algebraic colimit, over the finite dimensional subspaces $S \subset V$, of algebras of compactly supported continuous functions $\mathbb{R} \times S \to \mathrm{Cliff}(\mathbb{R} \oplus S)$. The map (4.8) restricts to an equivariant asymptotic homomorphism

$$\hat{\beta}_c : C_c(\mathbb{R}) - - > \mathcal{A}_c(V).$$

Using Higson-Kasparov's result, together with Lemma 4.3.3.13, we show in Proposition 4.4.10 that for the extension $\bar{\mathbb{E}}_X$ of Section 4.3, $\bar{\mathbb{E}}_X(\hat{\beta}_c)$ has a left homotopy inverse. Then in Corollary 4.4.12 we give the following application. Let $f : X \to Y$ be an equivariant map and let

$$\mathbb{E}_X \to \mathbb{E}_Y \tag{4.9}$$

be the natural transformation induced by f. Then

$$\mathbb{E}_X(\mathbb{C}) \to \mathbb{E}_Y(\mathbb{C})$$

is a weak equivalence whenever $\mathbb{E}_X(\mathcal{A}_c(V)) \to \mathbb{E}_Y(\mathcal{A}_c(V))$ is. In Section 4.5 we recall the notion of proper G-rings over a discrete homogeneous space G/H, introduced in [6], which is analogous to the same notion for C^*-algebras ([16]). It is shown in [16, Theorem 13.1] that the E-theory Baum-Connes assembly map for the full C^*-crossed product with coefficients in proper G-C^*-algebras is an isomorphism. The analogous result for algebraic K-theory of algebraic crossed products of groups and K-excisive $\mathbb{Q}$-algebras and the Farrell-Jones assembly map was proved in [6, Theorem 13.2.1]. Higson and Kasparov show in [20] that if the affine isometric action of G on V is metrically proper, then $\mathcal{A}_0(V)$ is a proper G-C^*-algebra. We prove in Theorem 4.5.14 that if

$$\tau : \mathbb{F} \to \mathbb{G}$$

is a natural transformation between functors G-$\mathfrak{BC}^* \to \mathrm{Spt}$, then the map $\tau(\mathcal{A}_c(V))$ is a weak equivalence whenever all the following conditions are satisfied:

- The action of G on V is metrically proper.
- The functors $\mathbb{F}$ and $\mathbb{G}$ satisfy excision and commute up to weak equivalence with filtering colimits along injective maps.
- If $H \subset G$ is a finite subgroup and P is proper over G/H, then $\tau(P)$ is an equivalence.

In Section 4.6 we use all these results to prove Theorem 4.2 (for general bornolocal C^*-algebras) and Corollaries 4.3 and 4.6; they are Theorem 4.6.1 and Corollaries 4.6.3 and 4.6.5, respectively.

These results were published in [9].

4.1 Bornolocal C^*-algebras

4.1.1 Definitions and examples

Let $(A, \|\|)$ be a normed $*$-algebra over $\mathbb{C}$ such that $\|a^*a\| = \|a\|^2$ for all $a \in A$. A *C^*-bornology* for A is a filtered family $\mathcal{F}$ of complete $*$-subalgebras that verifies $\bigcup_{\mathfrak{A}\in\mathcal{F}} \mathfrak{A} = A$. If $\mathcal{F}$ and $\mathcal{F}'$ are two C^*-bornologies on A, we say that $\mathcal{F}$ is *finer* than $\mathcal{F}'$ (and write $\mathcal{F} \prec \mathcal{F}'$) if for each $\mathfrak{A} \in \mathcal{F}$ there exists $\mathfrak{A}' \in \mathcal{F}'$ such that $\mathfrak{A} \subset \mathfrak{A}'$. If $\mathcal{F} \prec \mathcal{F}'$ and $\mathcal{F}' \prec \mathcal{F}$ we call the bornologies *equivalent*. A *bornolocal C^*-algebra* is a normed $*$-algebra A as above equipped with an equivalence class of C^*-bornologies. Thus a bornolocal C^*-algebra is a local C^*-algebra in the bornological sense (cf. [12, Definition 2.11]). We write $(A, \mathcal{F})$ or simply A for the algebra A equipped with the equivalence class of the C^*-bornology $\mathcal{F}$, depending on whether or not the latter needs to be emphasized. A *morphism* between two bornolocal C^*-algebras $(A, \mathcal{F})$ and $(B, \mathcal{G})$ is a $*$-homomorphism $f : A \to B$ such that $\mathcal{F} \prec f^{-1}(\mathcal{G}) := \{f^{-1}(\mathfrak{B}) : \mathfrak{B} \in \mathcal{G}\}$. Note that this definition depends only on the equivalence classes of the bornologies $\mathcal{F}$ and $\mathcal{G}$. For example if $(A, \mathcal{F})$ is a bornolocal C^*-algebra and $C \subset A$ is a closed $*$-subalgebra then C is again a bornolocal C^*-algebra with the *induced bornology*

$$\{\mathfrak{A} \cap C : \mathfrak{A} \in \mathcal{F}\} \tag{4.1.1.1}$$

and the inclusion is a homomorphism. We write $\mathfrak{BC}^*$ for the category of bornolocal C^*-algebras and morphisms.

Any C^*-algebra $\mathfrak{A}$ may be viewed as a bornolocal C^*-algebra with the trivial bornology $\mathcal{F} = \{\mathfrak{A}\}$. This gives a fully faithful embedding of the category of C^*-algebras into $\mathfrak{BC}^*$. If $\{A_i\}$ is a filtering system of bornolocal C^*-algebras with injective transfer maps then the algebraic colimit $A = \operatorname{colim}_i A_i$, equipped with the obvious colimit bornology, is the colimit of the system in $\mathfrak{BC}^*$. Thus any functor $F : C^*\text{-Alg} \to C^*\text{-Alg}$ which preserves monomorphisms extends to bornolocal C^*-algebras by

$$F(A, \mathcal{F}) = \operatorname*{colim}_{\mathcal{F}} F(\mathfrak{A}). \tag{4.1.1.2}$$

Hence, for example, if X is a locally compact (Hausdorff) space and $A \in \mathfrak{BC}^*$ then the algebra $C_0(X, A)$ of continuous functions vanishing at infinity is again in $\mathfrak{BC}^*$. Moreover the algebra of compactly supported continuous functions is also in $\mathfrak{BC}^*$, since we may write it as the colimit

$$C_c(X, A) = \operatorname{colim} \ker(C(K, \mathfrak{A}) \to C(\partial K, \mathfrak{A})).$$

Here the colimit runs over all pairs $(\mathfrak{A}, K)$ with $\mathfrak{A} \in \mathcal{F}$ and $K \subset X$ a compact subspace which is the closure of an open subset. Recall from [32, T.5.19] that the spatial tensor product $\underset{\sim}{\otimes}$ of injective morphisms of C^*-algebras is again injective. The *spatial tensor product* $A\underset{\sim}{\otimes}B$ of bornolocal C^*-algebras is defined by using (4.1.1.2) twice. For example, $C_c(X, A) = C_c(X)\underset{\sim}{\otimes}A$. The *graded spatial tensor product* $A\hat{\underset{\sim}{\otimes}}B$ of $\mathbb{Z}/2\mathbb{Z}$-graded bornolocal C^*-algebras A and B is defined similarly.

If $B \in \mathfrak{BC}^*$ we write $B[0,1] = C([0,1], B)$ for the algebra of continuous functions. Two homomorphisms $f_0, f_1 : A \to B \in \mathfrak{BC}^*$ are *homotopic* if there exists $H : A \to B[0,1] \in \mathfrak{BC}^*$ such that $\mathrm{ev}_i H = f_i$ $(i = 0, 1)$.

4.1.2 Exact sequences

If $(A,\mathcal{F}) \in \mathfrak{BC}^*$ then a *bornolocal ideal* in A is a ring theoretic, closed two-sided ideal I, equipped with the equivalence class of the induced bornology (4.1.1.1). Note that any such ideal is automatically self-adjoint. The kernel of a homomorphism $f : A \to B$ in $\mathfrak{BC}^*$ in the categorical sense is just the ring theoretic kernel $\ker f$ with the induced bornology. If $A = (A,\mathcal{F}) \in \mathfrak{BC}^*$ and $I \lhd A$ is a bornolocal ideal, then the cokernel of the inclusion map $I \subset A$ is A/I equipped with the equivalence class of the bornology $\{\mathfrak{A}/\mathfrak{A} \cap I : \mathfrak{A} \in \mathcal{F}\}$. A sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0 \tag{4.1.2.1}$$

of bornolocal C^*-algebras is *exact* if i is a kernel of p and p is a cokernel of i. By our previous remarks, if $B = (B,\mathcal{F})$ then (4.1.2.1) is isomorphic to the algebraic colimit of the exact sequences of C^*-algebras

$$0 \to A \cap \mathfrak{B} \xrightarrow{i} \mathfrak{B} \xrightarrow{p} \mathfrak{B}/A \cap \mathfrak{B} \to 0$$

with $\mathfrak{B} \in \mathcal{F}$, and this colimit coincides with the colimit in $\mathfrak{BC}^*$. Conversely, the colimit in $\mathfrak{BC}^*$ of any filtering system of short exact sequences of C^*-algebras along monomorphisms is exact.

4.2 Equivariant homology

4.2.1 Homotopy invariance, excision, stability and equivariant E-theory

Let G be a countable discrete group. Consider the category G-$\mathfrak{BC}^*$ of G-bornolocal C^*-algebras and equivariant homomorphisms. If $A, B \in G$-$\mathfrak{BC}^*$, we equip $A \underset{\sim}{\otimes} B$ with the diagonal action. Let $A[0,1] = A \underset{\sim}{\otimes} \mathbb{C}[0,1]$ with the trivial action on $\mathbb{C}[0,1]$. The natural map

$$c : A \to A[0,1], \quad c(a)(t) = a, \quad t \in [0,1],$$

is G-equivariant. Let $\mathbb{E} : G\text{-}\mathfrak{BC}^* \to \mathrm{Spt}$. We say that $\mathbb{E}$ is *homotopy invariant* if $\mathbb{E}(c)$ is a weak equivalence for every $A \in G$-$\mathfrak{BC}^*$. We say that $\mathbb{E}$ is *excisive* if for every exact sequence (4.1.2.1) of equivariant maps,

$$\mathbb{E}(A) \to \mathbb{E}(B) \to \mathbb{E}(C) \tag{4.2.1.1}$$

is a homotopy fibration sequence.

Any equivariant orthogonal decomposition $H = H_1 \perp H_2$ of a separable G-Hilbert space gives rise to a C^*-algebra homomorphism $\mathcal{K}(H_i) \to \mathcal{K}(H)$ $(i = 1,2)$ between the algebras of compact operators. We say that $\mathbb{E}$ is *G-stable* (resp. *stable*) if for every equivariant orthogonal decomposition as above (resp. for every decomposition as above where $\dim H_1 = 1$ and G acts trivially on H) and every $A \in G$-$\mathfrak{BC}^*$, $\mathbb{E}$ sends the maps

$$A \underset{\sim}{\otimes} \mathcal{K}(H_1) \to A \underset{\sim}{\otimes} \mathcal{K}(H) \leftarrow A \underset{\sim}{\otimes} \mathcal{K}(H_2) \tag{4.2.1.2}$$

to weak equivalences. Thus if H_1 and H_2 are G-Hilbert spaces and $\mathbb{E}$ is G-stable then the maps

(4.2.1.2) induce a weak equivalence

$$\mathbb{E}(A\underset{\sim}{\otimes}\mathcal{K}(H_1)) \xrightarrow{\sim} \mathbb{E}(A\underset{\sim}{\otimes}\mathcal{K}(H_2)).$$

4.2.2 Equivariant K-homology of stable algebras

In the next lemma and elsewhere, we shall use the *maximal tensor product* $C \otimes_\mu D$ of two bornolocal C^*-algebras C, D. If $(C, \mathcal{F})$, $(D, \mathcal{G}) \in \mathfrak{BC}^*$, this is the algebraic colimit

$$C \otimes_\mu D = \operatorname*{colim}_{\mathcal{F}\times\mathcal{G}} \mathfrak{C} \otimes_\mu \mathfrak{D}, \tag{4.2.2.1}$$

here $\mathfrak{C}\otimes_\mu\mathfrak{D}$ is the maximal tensor product of C^*-algebras (see Appendix A.2.2). One checks that the colimit depends only on the equivalence classes of $\mathcal{F}$ and $\mathcal{G}$, so that $C \otimes_\mu D$ is a well-defined $\mathbb{C}$-algebra. If either C or D is *nuclear*, i.e. it has a bornology in which every element is a nuclear C^*-algebra, then $C \otimes_\mu D = C\underset{\sim}{\otimes}D$ is the bornolocal C^*-algebra of Section 4.1.1. If G is a discrete group acting on both C and D, then we consider $C \otimes_\mu D$ as a G-algebra equipped with the diagonal action. If B, C and D are in G-$\mathfrak{BC}^*$ and C is nuclear, then there is an associativity isomorphism

$$(B \otimes_\mu C) \otimes_\mu D \cong B \otimes_\mu (C \otimes_\mu D). \tag{4.2.2.2}$$

In this case we shall abuse notation and write $B \otimes_\mu C \otimes_\mu D$ for $(B \otimes_\mu C) \otimes_\mu D$.

The following lemma is well-known and straightforward; it will be used in the proof of Theorem 4.2.2.4 below.

Lemma 4.2.2.3. *Let H be a G-Hilbert space; if $g \in G$, write $u_g \in \mathcal{B}(H)$ for the unitary implementing the action of g on H. Let I be a G-ring and $A, B \in G$-$\mathfrak{BC}^*$. Let $\underline{H}$ be H with the trivial G-action. Then the map*

$$\begin{aligned}(I \otimes (A \otimes_\mu \mathcal{K}(H) \otimes_\mu B)) \rtimes G &\to (I \otimes (A \otimes_\mu \mathcal{K}(\underline{H}) \otimes_\mu B)) \rtimes G\\ (x \otimes (a \otimes_\mu T \otimes_\mu b)) \rtimes g &\mapsto (x \otimes (a \otimes_\mu Tu_g \otimes_\mu b)) \rtimes g\end{aligned}$$

is an isomorphism.

We now come to the main theorem of this section.

Theorem 4.2.2.4. *Let G be a countable discrete group, I a G-ring, $B \in G$-$\mathfrak{BC}^*$ and $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ the algebra of compact operators with trivial G-action. Assume that I is K-excisive. Let X be a G-simplicial set. Then the functor*

$$G\text{-}\mathfrak{BC}^* \to \mathrm{Spt}, \quad A \mapsto H^G(X, K(I \otimes (A \otimes_\mu \mathcal{K} \otimes_\mu B)))$$

is excisive, homotopy invariant, and G-stable.

Proof. By [31, Corollary 10.4], C^*-algebras are K-excisive, and by [6, Proposition A.4.4] K-excisive rings are closed under filtering colimits. It follows that $A \otimes_\mu B$ is K-excisive for every pair of bornolocal C^*-algebras A and B. Hence $I \otimes (A \otimes_\mu B)$ is K-excisive for every $A, B \in \mathfrak{BC}^*$, by [6, Proposition A.5.3]. Besides, by [16, Lemma 4.1] and Section 4.1.2, $-\otimes_\mu B : \mathfrak{B}\mathfrak{C}^* \to \mathbb{C}\text{-Alg}$ is exact. Hence the functor of the proposition is excisive, by [6, Propositions 3.3.9 and 4.3.1]. Fix $n \in \mathbb{Z}$ and consider the functor

$$F : C^*\text{-Alg} \to \mathfrak{Ab}, \quad F(\mathfrak{C}) = H_n^G(X, K(I \otimes ((A \underset{\sim}{\otimes} \mathfrak{C}) \otimes_\mu B))).$$

Here $\mathfrak{C}$ is regarded as a G-C^*-algebra with trivial action. Again by [6, Propositions 3.3.9 and 4.3.1], F is split-exact. Hence $\mathfrak{C} \mapsto F(\mathfrak{C} \underset{\sim}{\otimes} \mathcal{K})$ is homotopy invariant, by Higson's homotopy invariance theorem [18, Theorem 3.2.2]. Specializing to $\mathfrak{C} = \mathbb{C}$, we obtain that the functor of the proposition is homotopy invariant, excisive and stable. To prove that it is also G-stable, it suffices to show that if $S \subset G$ is a subgroup, then

$$A \mapsto K((I \otimes (A \otimes_\mu \mathcal{K} \otimes_\mu B)) \rtimes \mathcal{G}(G/S))$$

is G-stable. By 1.3.8 and 1.3.9 there is a weak equivalence

$$K((I \otimes (A \otimes_\mu \mathcal{K} \otimes_\mu B)) \rtimes S) \xrightarrow{\sim} K((I \otimes (A \otimes_\mu \mathcal{K} \otimes_\mu B)) \rtimes \mathcal{G}(G/S)).$$

It is clear that $A \mapsto K((I \otimes (A \otimes_\mu \mathcal{K} \otimes_\mu B)) \rtimes S)$ is stable; by Lemma 4.2.2.3 it is also S-stable, and therefore G-stable. □

Remark 4.2.2.5. Theorem 4.2.2.4 will be used in full generality in the proof of Theorem 4.6.1. The application given in Corollary 4.2.2.7 below uses only the case $B = \mathbb{C}$.

Consider the comparison map

$$K \to KH \tag{4.2.2.6}$$

from algebraic K-theory to Weibel's homotopy algebraic K-theory [34].

Corollary 4.2.2.7. *Let X be a G-space. The map* (4.2.2.6) *induces a weak equivalence*

$$H^G(X, K(I \otimes (A \underset{\sim}{\otimes} \mathcal{K}))) \to H^G(X, KH(I \otimes (A \underset{\sim}{\otimes} \mathcal{K}))).$$

Proof. It suffices to show that the map of OrG-spectra

$$K((I \otimes (A \underset{\sim}{\otimes} \mathcal{K})) \rtimes \mathcal{G}(G/H)) \to KH((I \otimes (A \underset{\sim}{\otimes} \mathcal{K})) \rtimes \mathcal{G}(G/H))$$

is a weak equivalence. By 1.3.8 and 1.3.9 this is equivalent to proving that

$$K((I \otimes (A \underset{\sim}{\otimes} \mathcal{K})) \rtimes H) \to KH((I \otimes (A \underset{\sim}{\otimes} \mathcal{K})) \rtimes H)$$

is an equivalence for each subgroup $H \subset G$. By [34, Proposition 1.5], the map $K(R) \to KH(R)$ is an equivalence for K-regular R. Thus it suffices to show that $(I \otimes (A \underset{\sim}{\otimes} \mathcal{K})) \rtimes H$ is K-regular. By Theorem 4.2.2.4, the functor $K_*((I \otimes (- \underset{\sim}{\otimes} \mathcal{K})) \rtimes H)$ is homotopy invariant. It follows from this, using the argument of the proof of [30, Theorem 3.4], that $(I \otimes (A \underset{\sim}{\otimes} \mathcal{K})) \rtimes H$ is K-regular for every $A \in G$-$\mathfrak{B}\mathfrak{C}^*$. □

Remark 4.2.2.8. By [2, Remark 7.4], if J is any G-ring, there is a canonical weak equivalence

$$H^G(\mathcal{E}(G,\mathcal{F}in), KH(J)) \xrightarrow{\sim} H^G(\mathcal{E}(G,\mathcal{V}cyc), KH(J)).$$

Hence in view of Corollary 4.2.2.7, if I, A and $\mathcal{K}$ are as above, then the Farrell-Jones conjecture with coefficients in $J = I \otimes (A \underset{\sim}{\otimes} \mathcal{K})$ is equivalent to the isomorphism conjecture for the quadruple $(G, \mathcal{F}in, K, J)$.

4.3 Asymptotic morphisms

4.3.1 Basic definitions

We begin by recalling from [16] some basic definitions and facts concerning equivariant asymptotic morphisms of C^*-algebras. Let G be a discrete group and let $\mathfrak{B}$ be a G-C^*-algebra; write $C_b([1,\infty),\mathfrak{B})$ and $C_0([1,\infty),\mathfrak{B})$ for the G-C^*-algebras of bounded continuous functions and of continous functions vanishing at infinity, equipped with the induced actions. Consider the quotient

$$Q(\mathfrak{B}) = C_b([1,\infty),\mathfrak{B})/C_0([1,\infty),\mathfrak{B}). \tag{4.3.1.1}$$

If $n \geq 0$, we write Q^n for the n-fold composition of the functor Q. Let $\mathfrak{A}$ be another G-C^*-algebra. A G-equivariant *n-asymptotic homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ is a G-equivariant $*$-homomorphism $\mathfrak{A} \to Q^n(\mathfrak{B})$. Thus a 0-asymptotic morphism is the same as a homomorphism of G-C^*-algebras; 1-asymptotic morphisms are simply called *asymptotic morphisms.* We shall often write

$$f : \mathfrak{A} - - > \mathfrak{B}$$

for the equivariant morphism $f : \mathfrak{A} \to Q(\mathfrak{B})$. If $f : \mathfrak{A} - - > \mathfrak{B}$ is an equivariant asymptotic morphism then any set-theoretic lift $\phi : \mathfrak{A} \to C_b([1,\infty),\mathfrak{B})$ of f can be viewed as a bounded family of maps $\phi_t : \mathfrak{A} \to \mathfrak{B}$ varying continuously on $t \in [1,\infty)$ which, roughly speaking, tends to satisfy the conditons for an equivariant homomorphism as $t \to \infty$; see [16, Definitions 1.3 and 1.10] for details. Such a family is called an *equivariant asymptotic family* representing f; there is a one-to-one correspondence between equivariant asymptotic homomorphisms and classes of equivariant asymptotic families up to asymptotic equivalence [16, Proposition 1.11]. An *n-homotopy* between G-equivariant morphisms $f_0, f_1 : \mathfrak{A} \to Q^n(\mathfrak{B})$ is a G-equivariant morphism $H : \mathfrak{A} \to Q^n(\mathfrak{B}[0,1])$ such that $Q^n(\mathrm{ev}_i)H = f_i$ $(i = 0,1)$. By [16, Proposition 2.3], n-homotopy is an equivalence relation; we write $[[\mathfrak{A},\mathfrak{B}]]_n$ for the set of n-homotopy classes of n-asymptotic morphisms. Let $\pi : C_b([1,\infty),\mathfrak{A}) \to Q(\mathfrak{A})$ be the projection and let

$$\iota : \mathfrak{A} \to C_b([1,\infty),\mathfrak{A}), \quad \iota(a)(t) = a. \tag{4.3.1.2}$$

There is a map $[[\mathfrak{A},\mathfrak{B}]]_n \to [[\mathfrak{A},\mathfrak{B}]]_{n+1}$ sending the class of $f : \mathfrak{A} \to Q^n(\mathfrak{B})$ to that of $Q(f)\pi\iota : \mathfrak{A} \to Q^{n+1}(\mathfrak{B})$; we put

$$[[\mathfrak{A},\mathfrak{B}]] = \operatorname*{colim}_n [[\mathfrak{A},\mathfrak{B}]]_n.$$

If $\mathfrak{A}$ happens to be separable, then the map $[[\mathfrak{A},\mathfrak{B}]]_1 \to [[\mathfrak{A},\mathfrak{B}]]$ is bijective ([16, Theorem 2.16]). There is a category $\mathfrak{Q}_G$ whose objects are the G-C^*-algebras and where $\hom_{\mathfrak{Q}_G}(\mathfrak{A},\mathfrak{B}) = [[\mathfrak{A},\mathfrak{B}]]$

([16, Proposition 2.12]). The composite in $\mathfrak{Q}_G$ of the classes of $f : \mathfrak{A} \to Q^n(\mathfrak{B})$ and $g : \mathfrak{B} \to Q^m(\mathfrak{C})$ is the class of $Q^n(g)f$.

In the next section we shall need to consider equivariant asymptotic morphisms of bornolocal C^*-algebras. The definition is the same as in the C^*-algebra case; if A and $B \in G\text{-}\mathfrak{BC}^*$, a G-equivariant *asymptotic morphism* from A to B is a G-equivariant morphism

$$A \to Q(B) = C_b([1,\infty), B)/C_0([1,\infty), B).$$

Here $C_b([1,\infty), B)$ is the algebra of bounded continous functions with values in the normed algebra B. It is normed by the supremum norm, but has no obvious C^*-bornology; thus $Q(B)$ is just a normed G-$*$-algebra. As in the C^*-algebra case, equivariant asymptotic morphisms are in one-to-one correspondence with classes of equivariant asymptotic families up to asymptotic equivalence. The definition of 1-homotopy is also the same as in the C^*-algebra case. We do not consider n-asymptotic morphisms $A \to B$ for general $B \in G\text{-}\mathfrak{BC}^*$and $n \geq 2$.

4.3.2 Applying functors to asymptotic homomorphisms

We shall presently show that any excisive and homotopy invariant functor from G-C^*-algebras to spectra induces a functor $\mathfrak{Q}_G \to \mathrm{HoSpt}$ to the homotopy category. We begin by noting that the kernel $C_0([1,\infty), \mathfrak{B})$ of the projection π is equivariantly contractible. Hence if $\mathbb{E}$ is an excisive and homotopy invariant functor to spectra, then we have a natural map $\gamma_n : \mathbb{E}(Q^n(\mathfrak{B})) \to \mathbb{E}(\mathfrak{B})$ given by

$$\mathbb{E}(Q^n(\mathfrak{B})) \overset{\pi}{\underset{\sim}{\longleftarrow}} \mathbb{E}(C_b([1,\infty), Q^{n-1}(\mathfrak{B}))) \xrightarrow{\mathrm{ev}_1} \mathbb{E}(Q^{n-1}(\mathfrak{B})) \overset{\pi}{\underset{\sim}{\longleftarrow}} \dots \xrightarrow{\mathrm{ev}_1} \mathbb{E}(\mathfrak{B}).$$

Next observe that for $t \in [0,1]$ we have the commutative diagram below, where the vertical map in the middle is induced by $Q^n(\mathrm{ev}_t)$

$$\begin{tikzcd}
Q^{n+1}(\mathfrak{B}[0,1]) \arrow[d, "Q^{n+1}(\mathrm{ev}_t)"'] & C_b([1,\infty), Q^n(\mathfrak{B}[0,1])) \arrow[l, "\pi"'] \arrow[r, "\mathrm{ev}_1"] \arrow[d] & Q^n(\mathfrak{B}[0,1]) \arrow[d, "Q^n(\mathrm{ev}_t)"] \\
Q^{n+1}(\mathfrak{B}) & C_b([1,\infty), Q^n(\mathfrak{B})) \arrow[l, "\pi"'] \arrow[r, "\mathrm{ev}_1"] & Q^n(\mathfrak{B})
\end{tikzcd}$$

It follows that the maps $\gamma_n\mathbb{E}(Q^n(\mathrm{ev}_0))$ and $\gamma_n\mathbb{E}(Q^n(\mathrm{ev}_1)) : \mathbb{E}(Q^n(\mathfrak{B}[0,1])) \to \mathbb{E}(Q^n(\mathfrak{B}))$ represent the same map in HoSpt. Moreover, if $f : \mathfrak{A} \to Q^n(\mathfrak{B})$ is a homomorphism and ι is as in (4.3.1.2), then we have the following commutative diagram, where the middle horizontal map is induced by f

$$\begin{tikzcd}
Q(\mathfrak{A}) \arrow[r, "Q(f)"] & Q^{n+1}(\mathfrak{B}) & \\
C_b([1,\infty), \mathfrak{A}) \arrow[u, "\pi"] \arrow[r] & C_b([1,\infty), Q^n(\mathfrak{B})) \arrow[u, "\pi"] \arrow[r, "\mathrm{ev}_1"] & Q^n(\mathfrak{B}) \\
\mathfrak{A} \arrow[u, "\iota"] \arrow[r, "f"'] & Q^n(\mathfrak{B}) \arrow[u, "\iota"] \arrow[ur, equal] &
\end{tikzcd}$$

Hence the maps $\gamma_{n+1}\mathbb{E}(Q(f)(\pi\iota))$ and $\gamma_n(\mathbb{E}(f))$ are the same in the homotopy category. Thus we have a well-defined map

$$\begin{gathered}\bar{\mathbb{E}} : [[\mathfrak{A}, \mathfrak{B}]] \to \mathrm{HoSpt}(\mathbb{E}(\mathfrak{A}), \mathbb{E}(\mathfrak{B})) \\ (f : \mathfrak{A} \to Q^n(\mathfrak{B})) \mapsto \gamma_n \mathbb{E}(f).\end{gathered} \tag{4.3.2.1}$$

One checks further that the latter map is compatible with composition, so that we have a functor

$$\bar{\mathbb{E}} : \mathfrak{Q}_G \to \mathrm{HoSpt}.$$

Recall from [16, Theorem 6.9] that there is also an additive category E_G whose objects are the G-C^*-algebras and where the homomorphisms are given by

$$E_G(\mathfrak{A}, \mathfrak{B}) = [[\Sigma\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K} \underset{\sim}{\otimes} \mathcal{K}(\ell^2(G)), \Sigma\mathfrak{B} \underset{\sim}{\otimes} \mathcal{K} \underset{\sim}{\otimes} \mathcal{K}(\ell^2(G))]].$$

Here $\Sigma\mathfrak{A} = C_0(\mathbb{R}) \underset{\sim}{\otimes} \mathfrak{A}$ is the *suspension*. There is a functor

$$\mathfrak{Q}_G \to E_G \tag{4.3.2.2}$$

which is the identity on objects and on morphisms is induced by tensor product with $C_0(\mathbb{R}) \underset{\sim}{\otimes} \mathcal{K} \underset{\sim}{\otimes} \mathcal{K}(\ell^2(G))$ (see [16, Theorem 4.6]). We remark that if $\mathbb{E} : G\text{-}\mathfrak{C}^* \to \mathrm{Spt}$ is excisive, homotopy invariant, and G-stable, then

$$\mathbb{E}(\Sigma\mathfrak{A} \underset{\sim}{\otimes} \mathcal{K} \underset{\sim}{\otimes} \mathcal{K}(\ell^2(G))) \xrightarrow{\sim} \Omega\mathbb{E}(\mathfrak{A}).$$

Hence we can further extend $\bar{\mathbb{E}}$ to a functor

$$\bar{\bar{\mathbb{E}}} : E_G \to \mathrm{HoSpt}.$$

4.3.3 *The case of equivariant K-homology*

Let X be a G-space, $C \in G\text{-}\mathfrak{B}\mathfrak{C}^*$ and I an excisive G-ring. By Theorem 4.2.2.4, the functor

$$\mathbb{E} : G\text{-}\mathfrak{B}\mathfrak{C}^* \to \mathrm{Spt}, \quad \mathbb{E}(A) = H^G(X, K(I \otimes (A \otimes_\mu \mathcal{K} \otimes_\mu C))) \tag{4.3.3.1}$$

is excisive, homotopy invariant, and G-stable. Hence its restriction to G-C^*-algebras induces functors $\bar{\mathbb{E}} : \mathfrak{Q}_G \to \mathrm{HoSpt}$ and $\bar{\bar{\mathbb{E}}} : E_G \to \mathrm{HoSpt}$.

If we apply (4.3.2.1) to an equivariant homomorphism $f : \mathfrak{A} \to Q(\mathfrak{B})$, then for

$$\mathbb{F}(\mathfrak{A}) = H^G(X, K(I \otimes \mathfrak{A}))$$

we obtain the class of the composite

$$\begin{tikzcd}
\bar{\mathbb{E}}(f):\ \mathbb{F}(\mathfrak{A}\otimes_\mu \mathcal{K}\otimes_\mu C) \arrow[r, "\mathbb{F}(f\otimes_\mu 1)"] & \mathbb{F}(Q(\mathfrak{B})\otimes_\mu \mathcal{K}\otimes_\mu C) \\
\mathbb{F}(\mathfrak{B}\otimes_\mu \mathcal{K}\otimes_\mu C) & \mathbb{F}(C_b([1,\infty),\mathfrak{B})\otimes_\mu \mathcal{K}\otimes_\mu C). \arrow[l, "\mathbb{F}(\mathrm{ev}_1\otimes_\mu 1)"] \arrow[u, "\mathbb{F}(\pi\otimes_\mu 1)", "\wr"']
\end{tikzcd} \tag{4.3.3.2}$$

Next observe that for any G-C^*-algebra $\mathfrak{D}$, we have an equivariant map of exact sequences

$$\begin{tikzcd}
0 \to C_0([1,\infty),\mathfrak{B})\otimes_\mu \mathfrak{D} \arrow[r] \arrow[d, "\wr"] & C_b([1,\infty),\mathfrak{B})\otimes_\mu \mathfrak{D} \arrow[r] \arrow[d, "q"] & Q(\mathfrak{B})\otimes_\mu \mathfrak{D} \to 0 \arrow[d, "p"] \\
0 \to C_0([1,\infty),\mathfrak{B}\otimes_\mu \mathfrak{D}) \arrow[r] & C_b([1,\infty),\mathfrak{B}\otimes_\mu \mathfrak{D}) \arrow[r] & Q(\mathfrak{B}\otimes_\mu \mathfrak{D}) \to 0
\end{tikzcd} \tag{4.3.3.3}$$

If $\mathfrak{B}\in G$-C^* is nuclear and $(D,\mathcal{F})\in G$-$\mathfrak{B}\mathfrak{C}^*$ then $\mathfrak{B}\otimes_\mu D\in G$-$\mathfrak{B}\mathfrak{C}^*$, so $Q(\mathfrak{B}\otimes_\mu D)$ is defined. Taking the colimit of the diagrams (4.3.3.3) for $\mathfrak{D}\in\mathcal{F}$ we obtain a commutative diagram with exact rows

$$\begin{tikzcd}
0 \to C_0([1,\infty),\mathfrak{B})\otimes_\mu D \arrow[r] \arrow[d, "\wr"] & C_b([1,\infty),\mathfrak{B})\otimes_\mu D \arrow[r] \arrow[d, "q"] & Q(\mathfrak{B})\otimes_\mu D \to 0 \arrow[d, "p"] \\
0 \to C_0([1,\infty),\mathfrak{B}\otimes_\mu D) \arrow[r] & C_b([1,\infty),\mathfrak{B}\otimes_\mu D) \arrow[r] & Q(\mathfrak{B}\otimes_\mu D) \to 0
\end{tikzcd} \tag{4.3.3.4}$$

In particular $\mathbb{F}(\pi):\mathbb{F}(C_b([1,\infty),\mathfrak{B}\otimes_\mu\mathcal{K}\otimes_\mu C))\to\mathbb{F}(Q(\mathfrak{B}\otimes_\mu\mathcal{K}\otimes_\mu C))$ is a weak equivalence since $C_0([1,\infty),\mathfrak{B})$ is contractible and

$$\mathbb{F}(C_0([1,\infty),\mathfrak{B}\otimes_\mu\mathcal{K}\otimes_\mu C))\cong\mathbb{E}(C_0([1,\infty),\mathfrak{B})).$$

Hence we may also consider the composite

$$\begin{tikzcd}
\mathbb{F}(\mathfrak{A}\otimes_\mu \mathcal{K}\otimes_\mu C) \arrow[r, "\mathbb{F}(p(f\otimes_\mu 1))"] & \mathbb{F}(Q(\mathfrak{B}\otimes_\mu \mathcal{K}\otimes_\mu C)) \\
\mathbb{F}(\mathfrak{B}\otimes_\mu \mathcal{K}\otimes_\mu C) & \mathbb{F}(C_b([1,\infty),\mathfrak{B}\otimes_\mu \mathcal{K}\otimes_\mu C)). \arrow[l, "\mathbb{F}(\mathrm{ev}_1)"] \arrow[u, "\mathbb{F}(\pi)", "\wr"']
\end{tikzcd} \tag{4.3.3.5}$$

Lemma 4.3.3.6. *The maps* (4.3.3.2) *and* (4.3.3.5) *belong to the same class in* HoSpt.

Proof. Let $D=\mathcal{K}\otimes_\mu C$. The lemma follows from (4.3.3.4) and from the following commutative diagram

$$\begin{tikzcd}
C_b([1,\infty),\mathfrak{B})\otimes_\mu D \arrow[r, "ev_1\otimes_\mu 1"] \arrow[dr, "q"'] & \mathfrak{B}\otimes_\mu D \\
& C_b([1,\infty),\mathfrak{B}\otimes_\mu D). \arrow[u, "\mathrm{ev}_1"']
\end{tikzcd}$$

□

The following proposition summarizes our previous discussion.

Proposition 4.3.3.7. *Let G be a discrete group, X a G-space, $C \in G$-$\mathfrak{BC}^*$, and I a K-excisive G-ring. Consider the functors*

$$\begin{gathered}\mathbb{F} : G\text{-Alg} \to \text{Spt}, \\ \mathbb{F}(A) = H^G(X, K(I \otimes A)), \\ \mathbb{E} : G\text{-}\mathfrak{BC}^* \to \text{Spt}, \\ \mathbb{E}(A) = \mathbb{F}(A \otimes_\mu \mathcal{K} \otimes_\mu C).\end{gathered}$$

Then the restriction of $\mathbb{E}$ to the category of G-C^*-algebras induces functors $\bar{\mathbb{E}} : \mathfrak{Q}_G \to \text{HoSpt}$ and $\bar{\bar{\mathbb{E}}} : E_G \to \text{HoSpt}$ from the equivariant asymptotic category and equivariant E-theory to the homotopy category of spectra. The diagram

$$\begin{tikzcd} & E_G \arrow[dr, "\bar{\bar{\mathbb{E}}}"] & \\ \mathfrak{Q}_G \arrow[ur] \arrow[rr, "\bar{\mathbb{E}}"'] & & \text{HoSpt} \\ G\text{-}\mathfrak{C}^* \arrow[u] \arrow[rr, "\mathbb{E}"'] & & \text{Spt} \arrow[u] \end{tikzcd}$$

commutes up to natural equivalence. If $f : \mathfrak{A} - -- > \mathfrak{B}$ is an equivariant asymptotic homomorphism, then $\bar{\mathbb{E}}(f)$ is the homotopy class of the composite of diagram (4.3.3.2)*. If moreover $\mathfrak{B}$ is nuclear, then the class of the latter map is the same as that of the composite of diagram* (4.3.3.5)*.*

Proof. We showed in Section 4.3.2 that any excisive and homotopy invariant functor G-$\mathfrak{C}^* \to \text{Spt}$ extends to a functor $\mathfrak{Q}_G \to \text{HoSpt}$, and moreover to $E_G \to \text{HoSpt}$ if in addition the functor is G-stable. By Theorem 4.2.2.4, this applies to the functor $\mathbb{E}$. The equivalence between the maps (4.3.3.2) and (4.3.3.5) is established by Lemma 4.3.3.6. □

Example 4.3.3.8. Let $\mathfrak{A}$ be a C^*-algebra. For $a \in C_0(\mathbb{R}, \mathfrak{A})$ and $t \in [1, \infty)$, put

$$\phi_0 : C_0(\mathbb{R}, \mathfrak{A}) \to C_b([1, \infty), C_0(\mathbb{R}, \mathfrak{A})), \quad \phi_0(a)(t)(x) = a(x/t). \tag{4.3.3.9}$$

Let $\mathfrak{B}$ be another C^*-algebra and let

$$f_0 : C_0(\mathbb{R}, \mathfrak{A}) \to \mathfrak{B}$$

be a $*$-homomorphism. Consider the map

$$\hat{f}_0 : C_0(\mathbb{R}, \mathfrak{A}) \to Q(\mathfrak{B}), \quad \hat{f}_0(a) = \pi(f_0 \phi_0(a)).$$

Assume that $\mathfrak{B}$ is nuclear, and let $C \in \mathfrak{BC}^*$. Then under the isomorphism

$$C_0(\mathbb{R}, \mathfrak{A}) \otimes_\mu C \cong C_0(\mathbb{R}, \mathfrak{A} \otimes_\mu C),$$

the map $p(\hat{f}_0 \otimes_\mu 1)$ identifies with $\widehat{f_0 \otimes_\mu 1}$. Thus if $\mathfrak{A}, \mathfrak{B} \in G$-$\mathfrak{C}^*$, $C \in G$-$\mathfrak{BC}^*$, and $\hat{f}_0$ is G-

equivariant, then, writing 1 for the identity map of $\mathcal{K}\otimes_\mu C$, we have that $\widehat{f_0\otimes_\mu 1}$ is G-equivariant, and $\bar{\mathbb{E}}(\hat{f}_0)$ is equivalent to the composite

$$\begin{tikzcd} \mathbb{F}(C_0(\mathbb{R},\mathfrak{A}\otimes_\mu\mathcal{K}\otimes_\mu C)) \arrow[r, "\mathbb{F}(\widehat{f_0\otimes_\mu 1})"] & \mathbb{F}(Q(\mathfrak{B}\otimes_\mu\mathcal{K}\otimes_\mu C)) \\ \mathbb{F}(\mathfrak{B}\otimes_\mu\mathcal{K}\otimes_\mu C) & \mathbb{F}(C_b([1,\infty),\mathfrak{B}\otimes_\mu\mathcal{K}\otimes_\mu C)). \arrow[l, "\mathbb{F}(\mathrm{ev}_1)"] \arrow[u, "\mathbb{F}(\pi)", "\wr"'] \end{tikzcd} \tag{4.3.3.10}$$

Now let $A, B, C \in G\text{-}\mathfrak{BC}^*$ with B nuclear. Formula (4.3.3.9) defines a homomorphism $C_0(\mathbb{R},A)\to C_b([1,\infty),C_0(\mathbb{R},A))$, which restricts to

$$\phi_c : C_c(\mathbb{R},A)\to C_b([1,\infty),C_c(\mathbb{R},A)).$$

Let $\#\in\{0,c\}$ and let $f_\# : C_\#(\mathbb{R},A)\to B$ be a $*$-homomorphism. Put

$$\hat{f_\#} : C_\#(\mathbb{R},A)\to Q(B),\quad \hat{f_\#}(a)=\pi(f_\#\phi_\#(a)). \tag{4.3.3.11}$$

Assume that $\hat{f_\#}$ is G-equivariant; write 1 for the identity map of $\mathcal{K}\otimes_\mu C$. Then $\widehat{f_\#\otimes_\mu 1}$ is again G-equivariant. Moreover, by Proposition 1.3.12 and Theorem 4.2.2.4, the map

$$\mathbb{F}(\pi):\mathbb{F}(C_b([1,\infty),B\otimes_\mu\mathcal{K}\otimes_\mu C))\to\mathbb{F}(Q(B\otimes_\mu\mathcal{K}\otimes_\mu C))$$

is a weak equivalence. Let $\bar{\mathbb{E}}(\hat{f_\#})$ be the composite

$$\begin{tikzcd} \bar{\mathbb{E}}(\hat{f_\#}):\ \mathbb{F}(C_\#(\mathbb{R},A\otimes_\mu\mathcal{K}\otimes_\mu C)) \arrow[r, "\mathbb{F}(\widehat{f_\#\otimes_\mu 1})"] & \mathbb{F}(Q(B\otimes_\mu\mathcal{K}\otimes_\mu C)) \\ \mathbb{F}(B\otimes_\mu\mathcal{K}\otimes_\mu C) & \mathbb{F}(C_b([1,\infty),B\otimes_\mu\mathcal{K}\otimes_\mu C)). \arrow[l, "\mathbb{F}(\mathrm{ev}_1)"] \arrow[u, "\mathbb{F}(\pi)", "\wr"'] \end{tikzcd} \tag{4.3.3.12}$$

In the next section we shall need the following trivial observation.

Lemma 4.3.3.13. *Let $i: A\to A'$, $j: B\to B'\in G\text{-}\mathfrak{BC}^*$ and let $f_c: C_c(\mathbb{R},A)\to B$ and $f_0: C_0(\mathbb{R},A')\to B'$ be $*$-homomorphisms. Assume that B and B' are nuclear and that the diagram*

$$\begin{tikzcd} C_0(\mathbb{R},A') \arrow[r, "f_0"] & B' \\ C_c(\mathbb{R},A) \arrow[u, "i"] \arrow[r, "f_c"'] & B \arrow[u, "j"'] \end{tikzcd}$$

commutes. Further assume that $\hat{f}_0$ and $\hat{f}_c$ are G-equivariant. Let $\mathbb{E}$ be as in Proposition 4.3.3.7 *and let* $\bar{\mathbb{E}}(\hat{f_\#})$ *be as in* (4.3.3.12) *(*$\# \in \{0, c\}$*). Then the diagram*

$$\begin{tikzcd}
\mathbb{E}(C_0(\mathbb{R}, A')) \arrow[r, "\bar{\mathbb{E}}(\hat{f}_0)"] & \mathbb{E}(B') \\
\mathbb{E}(C_c(\mathbb{R}, A)) \arrow[u, "\mathbb{E}(i)"] \arrow[r, "\bar{\mathbb{E}}(\hat{f}_c)"'] & \mathbb{E}(B) \arrow[u, "\mathbb{E}(j)"']
\end{tikzcd}$$

is homotopy commutative.

4.4 A Dual Dirac element

The purpose of this section is to prove a compactly supported variant of a theorem of Higson and Kasparov ([20, Theorem 6.10]). We start by recalling some material from [19], [20], and [21]. A *Euclidean space* is a real pre-Hilbert space. Let V be a countably infinite dimensional Euclidean space. Write $\mathcal{F}(V)$ for the set of finite dimensional affine subspaces of V. For $S \in \mathcal{F}(V)$ put $S^0 = \{s_1 - s_2 : s_i \in S\}$.

Write $\mathrm{Cliff}(S)$ for the complexified Clifford algebra of S^0, that is, the universal complex algebra with unit which contains S^0 as a real linear subspace in such a way that $s^2 = \|s\|^2 1$, for every s in S^0. If $\{s_1, ..., s_n\}$ is an orthonormal basis for S^0 then the monomials $s_{i_1}...s_{i_k}$ for $i_1 < ... < i_k$ form a linear basis for $\mathrm{Cliff}(S)$. We give $\mathrm{Cliff}(S)$ a Hermitian inner product by assuming these monomials to be orthonormal; the inner product does not depend on a choice of basis. By thinking of the algebra $\mathrm{Cliff}(S)$ as acting by left multiplication on the *Hilbert space* $\mathrm{Cliff}(S)$, we endow the Clifford algebra with the structure of a C^*-algebra, in such a way that each $s \in S^0$ is self adjoint. It is a $\mathbb{Z}/2\mathbb{Z}$- graded C^*-algebra, with each $s \in S^0$ having grading degree one.

For $\# \in \{c, 0\}$, put

$$\mathcal{C}_\#(S) = C_\#(S, \mathrm{Cliff}(S)).$$

Observe that the grading on $\mathrm{Cliff}(S)$ induces one on $\mathcal{C}_\#(S)$. For example $\mathrm{Cliff}(\mathbb{R}) = \mathbb{C} \oplus u\mathbb{C}$ where u is a degree one element satisfying $u^2 = 1$. Thus

$$\mathcal{C}_\#(\mathbb{R}) = C_\#(\mathbb{R}) \oplus uC_\#(\mathbb{R}).$$

We regard $\mathcal{C}_\#(\mathbb{R})$ as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra with homogeneous components $\mathcal{C}_\#(\mathbb{R})_j = u^j C_\#(\mathbb{R})$ $(j = 0, 1)$. In addition the algebra $C_\#(\mathbb{R})$ is also $\mathbb{Z}/2\mathbb{Z}$-graded according to even and odd functions. For $f \in C_\#(\mathbb{R})$ write $f = f^{\mathrm{even}} + f^{\mathrm{odd}}$ for its even-odd decomposition. One checks that the map

$$\theta : C_\#(\mathbb{R}) \to \mathcal{C}_\#(\mathbb{R}), \quad \theta(f) = f^{\mathrm{even}} + uf^{\mathrm{odd}} \tag{4.4.1}$$

is a homogeneous isometric embedding. Let $X \in C(\mathbb{R})$ be the identity function. We may interpret θ as the functional calculus of the degree one, essentially self-adjoint, unbounded operator of multiplication by $Xu \in C(\mathbb{R}, \mathrm{Cliff}(\mathbb{R}))$; we have

$$\theta(f) = f(Xu). \tag{4.4.2}$$

We will identify $C_\#(\mathbb{R}) = \theta(C_\#(\mathbb{R}))$. Consider the graded spatial tensor product

$$\mathcal{A}_\#(S) = C_\#(\mathbb{R}) \hat{\underset{\sim}{\otimes}} \mathcal{C}_\#(S). \tag{4.4.3}$$

Using the identification above, we may regard $\mathcal{A}_\#(S)$ as a subalgebra of

$$C_\#(\mathbb{R} \times S, \mathrm{Cliff}(\mathbb{R} \oplus S^0)).$$

We have

$$\mathcal{A}_\#(S) = \{f = f^0 + uf^1 \in C_\#(\mathbb{R} \times S, \mathrm{Cliff}(\mathbb{R} \oplus S^0)) : f^i(-t, s) = (-1)^i f^i(t, s)\}.$$

If $S_1 \subset S_2 \in \mathcal{F}(V)$, define $S_{21} = S_2^0 \ominus S_1^0$ and write $S_2 = S_1 + S_{21}$. Then $\mathcal{A}_\#(S_2) = \mathcal{A}_\#(S_{21}) \hat{\underset{\sim}{\otimes}} \mathcal{C}_\#(S_1)$. Following [21], we write $C_{21} : S_{21} \to \mathrm{Cliff}(S_{21})$ for the inclusion and $X \in C(\mathbb{R})$ for the identity function, considered as degree one, essentially self-adjoint, unbounded multipliers of $\mathcal{C}_0(S_{21})$ and $C_0(\mathbb{R})$, with domains $\mathcal{C}_c(S_{21})$ and $C_c(\mathbb{R})$. Using functional calculus, one obtains a map

$$\beta_{21} : \mathcal{A}_0(S_1) \to \mathcal{A}_0(S_2), \qquad \beta_{21}(f \hat{\underset{\sim}{\otimes}} g) = f(X \hat{\underset{\sim}{\otimes}} 1 + 1 \hat{\underset{\sim}{\otimes}} C_{21}) \hat{\underset{\sim}{\otimes}} g. \tag{4.4.4}$$

Lemma 4.4.5. *Let $v \in S_1 \subset S_2 \in \mathcal{F}(V)$, $\rho > 0$, $f \in \mathcal{A}_c(S_1)$ with $\mathrm{supp}(f) \subset D^1((0, v), \rho)$, the closed ball in $\mathbb{R} \times S_1$. Then $\mathrm{supp}(\beta_{21}(f)) \subset D^2((0, v), \rho)$, the closed ball in $\mathbb{R} \times S_2$. In particular the map (4.4.4) sends $\mathcal{A}_c(S_1)$ to $\mathcal{A}_c(S_2)$.*

Proof. It follows from the fact that if s_2 decomposes as $s_2 = s_1 + s_{21} \in S_1 + S_{21}$ then

$$\beta_{21}(f)(x, s_2) = f(xu + s_{21}, s_1), \tag{4.4.6}$$

and that for each x, the spectrum of $xu + s_{21}$ is $\{\pm\sqrt{x^2 + ||s_{21}||^2}\}$. □

Remark 4.4.7. It follows from (4.4.1), (4.4.2), and (4.4.6), that the map (4.4.4) is injective.

By [21, Proposition 3.2], if $S_1 \subset S_2 \subset S_3$, then $\beta_{31} = \beta_{32}\beta_{21}$. Let $\mathcal{A}_0(V)$ be the C^*-algebra colimit of the direct system $\{\beta_{TS} : \mathcal{A}_0(S) \to \mathcal{A}_0(T)\}$. Also let

$$\mathcal{A}_c(V) = \operatorname*{colim}_{\mathcal{F}(V)} \mathcal{A}_c(S)$$

be the algebraic colimit; by Remark 4.4.7 this is the colimit in $\mathfrak{BC}^*$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{A}_0(0) \arrow[r, "\beta_0"] & \mathcal{A}_0(V) \\ \mathcal{A}_c(0) \arrow[u] \arrow[r, "\beta_c"'] & \mathcal{A}_c(V) \arrow[u] \end{tikzcd} \tag{4.4.8}$$

Now let G be a discrete group acting on V by affine isometries. Then for each $g \in G$ there are an orthogonal transformation $\ell(g)$ and a vector $\tau(g) \in V$ such that

$$g \cdot v = \ell(g)(v) + \tau(g) \qquad (v \in V). \tag{4.4.9}$$

The G-action on V induces an action on $\mathcal{A}_\#(V)$ defined as follows

$$(g\cdot f)(v)=\ell(g)(f(g^{-1}\cdot v)).$$

We regard $\mathcal{A}_\#(0)$ and $\mathcal{A}_\#(V)$ as G-algebras with the trivial and the induced action, respectively.

In general, the map $\beta_\#:\mathcal{A}_\#(0)\to\mathcal{A}_\#(V)$ is not G-equivariant; however this can be fixed asymptotically. Indeed the asymptotic homorphism

$$\hat{\beta}_\#:\mathcal{A}_\#(0)--->\mathcal{A}_\#(V)$$

defined by (4.3.3.11) is G-equivariant.

The following proposition is an immediate consequence of a theorem of Higson and Kasparov and of the results of the previous section.

Proposition 4.4.10. *(cf. [20, Theorem 6.8]). Let G be a countable discrete group acting on V by affine isometries. Let X be a G-space, let I be a K-excisive G-ring, and let $B\in G\text{-}\mathfrak{BC}^*$. Consider the functor*

$$\mathbb{E}_X:G\text{-}\mathfrak{BC}^*\to\mathrm{Spt},\quad \mathbb{E}_X(A)=H^G(X,K(I\otimes(A\otimes_\mu\mathcal{K}\otimes_\mu B))).$$

Then the map $\bar{\mathbb{E}}_X(\hat{\beta}_c)$ defined in (4.3.3.12) *is a split monomorphism in* HoSpt.

Proof. Put $\mathbb{E}=\mathbb{E}_X:G\text{-}\mathfrak{BC}^*\to\mathrm{Spt}$. By Proposition 4.3.3.7 and [20, Theorems 6.8 and 6.11], the functor (4.3.2.2) sends the class of $\hat{\beta}_0$ to an isomorphism in E_G. Hence in view of (4.4.8) and of Lemma 4.3.3.13 it suffices to show that $\mathbb{E}$ sends the inclusion $C_c(\mathbb{R})=\mathcal{A}_c(0)\to\mathcal{A}_0(0)=C_0(\mathbb{R})$ to a weak equivalence. Because $\mathbb{E}$ commutes up to weak homotopy equivalence with filtering colimits along injective maps, the natural map $\mathrm{colim}_{\rho>0}\mathbb{E}(C_0(-\rho,\rho))\to\mathbb{E}(C_c(\mathbb{R}))$ is a weak equivalence. For each $\rho>0$, $C_0(-\rho,\rho)\lhd C_0(\mathbb{R})$ is an ideal and the quotient $C_0(\mathbb{R})/C_0(-\rho,\rho)\cong C_0((-\infty,-\rho]\cup[\rho,\infty))$ is contractible; indeed $H(f)(s,t)=f(t/s)$ is a contraction. Thus because the functor $\mathbb{E}$ is homotopy invariant and excisive, $\mathbb{E}(C_0(-\rho,\rho))\to\mathbb{E}(C_0(\mathbb{R}))$ is a weak equivalence. Hence we have a weak equivalence

$$\mathbb{E}(C_c(\mathbb{R}))\xrightarrow{\sim}\mathbb{E}(C_0(\mathbb{R})).\tag{4.4.11}$$

This finishes the proof. □

Corollary 4.4.12. *Let Y be another G-space, and $f:X\to Y$ an equivariant map. Let $\tau:\mathbb{E}_X\to\mathbb{E}_Y$ be the natural map induced by f. Assume that $\tau(\mathcal{A}_c(V))$ is a weak equivalence. Then $\tau(\mathbb{C})$ is a weak equivalence too.*

Proof. By excision and homotopy invariance, $\tau(\mathbb{C})$ is equivalent to the delooping of $\tau(C_0(\mathbb{R}))$ in HoSpt. By (4.4.11) the latter map is equivalent to $\tau(C_c(\mathbb{R}))$. The corollary now follows from the proposition above and the fact that a retract of an isomorphism is an isomorphism. □

4.5 *Proper actions*

Let G be a discrete group. If $J\in G$-Rings is commutative but not necessarily unital and $I\in G$-Rings, then by a *compatible (G,J)-algebra structure* on I we understand a J-bimodule structure on I such that the following identities hold for $a, b \in I$, $c \in J$, and $g \in G$:

$$\begin{aligned} c \cdot a &= a \cdot c, \\ c \cdot (ab) &= (c \cdot a)b = a(c \cdot b), \\ g(c \cdot a) &= g(c) \cdot g(a). \end{aligned} \tag{4.5.1}$$

If I and J are $*$-$\mathbb{C}$-algebras we will additionally require the following two conditions

$$(\lambda c) \cdot a = c \cdot (\lambda a), \quad (c \cdot a)^* = c^* \cdot a^* \quad (\lambda \in \mathbb{C}, \quad c \in J, \quad a \in I). \tag{4.5.2}$$

If moreover I and J are normed, we will further ask that

$$||c \cdot a|| \leq ||c||||a||, \quad (c \in J, \quad a \in I). \tag{4.5.3}$$

We say that a compatible (G, J)-algebra structure is *full* if it satisfies the additional condition

$$J \cdot I = I. \tag{4.5.4}$$

If $(A, \mathcal{F}), (B, \mathcal{G}) \in G$-$\mathfrak{BC}^*$ with A commutative, for a compatible (G, A)-algebra structure on B to be *full* we shall also require that $\mathcal{G}$ be equivalent to the filtration $\mathcal{F} \cdot \mathcal{G}$ consisting of the $*$-subalgebras $\mathfrak{A} \cdot \mathfrak{B}$ with $\mathfrak{A} \in \mathcal{F}$ and $\mathfrak{B} \in \mathcal{G}$:

$$\mathcal{F} \cdot \mathcal{G} \sim \mathcal{G}. \tag{4.5.5}$$

Let $H \subset G$ be a subgroup. The ring

$$\mathbb{Z}^{(G/H)} = \{f : G/H \to \mathbb{Z} : |\mathrm{supp}(f)| < \infty\} = \bigoplus_{gH \in G/H} \mathbb{Z}$$

has a natural G-action. We say that a G-ring I is *proper* over G/H if it carries a full compatible $(G, \mathbb{Z}^{(G/H)})$-structure. Observe that

$$\mathbb{C}^{(G/H)} = C_c(G/H) \in G\text{-}\mathfrak{BC}^* \tag{4.5.6}$$

is the algebra of compactly supported continuous functions. We say that $A \in G$-$\mathfrak{BC}^*$ is *proper* over G/H if it carries a full compatible $(G, \mathbb{C}^{(G/H)})$-algebra structure. Then if $x \in G/H$ and χ_x is the characteristic function, (4.5.2) implies that multiplication by χ_x is a $*$-homomorphism $A \to A$ with image $A_x = \chi_x A$. Hence A_x is a closed $*$-subalgebra, and we have a direct sum decomposition

$$A = \bigoplus_{x \in G/H} A_x \tag{4.5.7}$$

where each $A_x \in \mathfrak{BC}^*$, and $A_H \in H$-$\mathfrak{BC}^*$. If $\mathcal{F}$ is a bornology in the equivalence class of A, then the induced C^*-bornology in A_x consists of the C^*-algebras $\mathfrak{A}_x = \mathfrak{A} \cap A_x$ with $\mathfrak{A} \in \mathcal{F}$. The algebra A also carries the following C^*-bornology

$$\mathcal{F}^\bullet = \{\bigoplus_{x \in F} \mathfrak{A}_x : F \subset G/H \text{ finite}, \mathfrak{A} \in \mathcal{F}\}.$$

Condition (4.5.5) implies that $\mathcal{F}^\bullet$ is equivalent to $\mathcal{F}$:

$$\mathcal{F} \sim \mathcal{F}^\bullet. \tag{4.5.8}$$

Remark 4.5.9. By (4.5.7) and (4.5.8), if G/H is infinite and A is nonzero and proper over G/H, then A cannot be complete, since it is isomorphic as a bornolocal C^*-algebra to an infinite algebraic direct sum of copies of A_H. In particular, no nonzero G-C^*-algebra can be proper in our sense over an infinite homogeneous space G/H.

Lemma 4.5.10. *Let $A, B \in G\text{-}\mathfrak{BC}^*$. Assume that A is proper over G/H. Then $A \widetilde{\otimes} B$ and $A \otimes_\mu B$ are proper over G/H, as a G-bornolocal C^*-algebra in the first case, and as a G-$*$-algebra in the second.*

Proof. Straightforward. □

Lemma 4.5.11. *Let $A, B \in G\text{-}\mathfrak{BC}^*$ with A commutative and let $H \subset G$ be a subgroup. Assume that A is proper over G/H and that B is equipped with a full compatible (G, A)-algebra structure. Then B is proper over G/H.*

Proof. For $x \in G/H$ let $B_x = A_x B$. It follows from (4.5.7) that $B = \sum_x B_x$. Next we show that $B_x \cap B_y = 0$ if $x \neq y$. Let $b \in B_x \cap B_y$. Then there exist n, $a_1, \ldots, a_n \in A_x$ and $b_1, \ldots, b_n \in B$ such that

$$b = \sum_{i=1}^{n} a_i b_i. \tag{4.5.12}$$

Because A_x is a bornolocal C^*-algebra, there is a C^*-subalgebra $\mathfrak{A}_x \subset A_x$, such that $a_1, \ldots, a_n \in \mathfrak{A}_x$. Let $\{e_\lambda\}$ be a bounded approximate unit in $\mathfrak{A}_x$. Use (4.5.3) and (4.5.12) to show that $\lim_\lambda e_\lambda b = b$. On the other hand, $e_\lambda \in A_x$ and $b \in B_y$ implies $e_\lambda b = 0$. Hence $B_x \cap B_y = 0$, as claimed. Define an action of $\mathbb{C}^{(G/H)}$ on B as follows. For $c = \sum_x \lambda_x \chi_x \in \mathbb{C}^{(G/H)}$ and $b = \sum_x b_x \in B$, put

$$c \cdot b = \sum_x \lambda_x b_x.$$

One checks that this action satisfies (4.5.1), (4.5.2) and (4.5.3). Moreover (4.5.5) and (4.5.8) together imply that if $B = (B, \mathcal{G})$ then $\mathcal{G}^\bullet \sim \mathcal{G}$. Thus B is proper over G/H. □

Let G be a countable discrete group and V a Euclidean space of countably infinite dimension where G acts by affine isometries. We say that the action of G on V is *metrically proper* if

$$\lim_{g \to \infty} \|g \cdot v\| = \infty \qquad (\forall v \in V). \tag{4.5.13}$$

The condition that a group G admits such an action is the *Haagerup approximation property*. In the literature, the groups that have this property are sometimes called *a-T-menable groups* and sometimes *Haagerup groups*.

The purpose of this section is to prove the following.

Theorem 4.5.14. *Let G be a countable discrete group and let V be a Euclidean space of countably infinite dimension with an action of G by affine isometries. Let $\mathbb{E}, \mathbb{F} : G\text{-}\mathfrak{BC}^* \to \mathrm{Spt}$ be functors and $\tau : \mathbb{E} \to \mathbb{F}$ a natural transformation. Assume:*

i) The action of G on V is metrically proper.

ii) If $H \subset G$ is a finite subgroup and $P \in G\text{-}\mathfrak{BC}^*$ is proper over G/H, then $\tau(P)$ is a weak equivalence.

iii) The functors $\mathbb{E}$ and $\mathbb{F}$ are excisive and commute with filtering colimits along injective maps up to weak equivalence.

Then the map $\tau(\mathcal{A}_c(V))$ is a weak equivalence.

The proof of Theorem 4.5.14 will be given at the end of the section. First we need to introduce some notation and prove some lemmas. Let $S \in \mathcal{F}(V)$; consider the subalgebra

$$\mathcal{Z}_c(S) = \{f \in C_c(\mathbb{R} \times S) : f(-t, s) = f(t, s)\} \subset \mathcal{A}_c(S).$$

Observe that $\mathcal{Z}_c(S)$ lies in the center of $\mathcal{A}_c(S)$ and, moreover, we have

$$\mathcal{A}_c(S) = \mathcal{Z}_c(S)\mathcal{A}_c(S). \tag{4.5.15}$$

Write

$$\mathbb{R}_+ = [0, \infty).$$

Restriction along the inclusion $\mathbb{R}_+ \times S \subset \mathbb{R} \times S$ induces an isomorphism

$$\mathcal{Z}_c(S) \cong C_c(\mathbb{R}_+ \times S). \tag{4.5.16}$$

From now on we shall identify both sides of (4.5.16). Let $S \subset T \in \mathcal{F}(V)$; every element of T writes uniquely as

$$t = \pi_S(t) + \pi_S^\perp(t) \quad \pi_S(t) \in S, \quad \pi_S^\perp(t) \in T^0 \ominus S^0.$$

Consider the map

$$\begin{aligned} p_{ST} &: \mathbb{R}_+ \times T \to \mathbb{R}_+ \times S, \\ p_{ST}(x, t) &= (\sqrt{x^2 + ||\pi_S^\perp(t)||^2}, \pi_S(t)). \end{aligned} \tag{4.5.17}$$

In view of (4.4.6), under the isomorphism of (4.5.16), the restriction of β_{TS} to $\mathcal{Z}_c(S)$ identifies with composition with p_{ST}:

$$\beta_{TS}(f) = f p_{ST}. \tag{4.5.18}$$

Put

$$\mathcal{Z}_c(V) = \operatorname*{colim}_S \mathcal{Z}_c(S).$$

Consider the inverse system of locally compact topological spaces and proper maps $\{p_{ST} : \mathbb{R}_+ \times T \to \mathbb{R}_+ \times S\}$. Let $\mathfrak{H} = \overline{V}$ be the Hilbert space completion; write $\mathfrak{H}_w$ for $\mathfrak{H}$ equipped with the locally convex topology of weak convergence. Equip

$$\mathfrak{X} := \mathbb{R}_+ \times \mathfrak{H} \tag{4.5.19}$$

with the coarsest topology such that both the projection $\mathbb{R}_+ \times \mathfrak{H} \to \mathfrak{H}_w$ and the map $\mathbb{R}_+ \times \mathfrak{H} \to \mathbb{R}_+$, $(x, h) \mapsto \sqrt{x^2 + ||h||^2}$ are continuous. If $h \in \mathfrak{H}$ and $S \in \mathcal{F}(V)$, write $h = \pi_S(h) + \pi_S^\perp(h) \in$

$S + S_0^\perp$. Let

$$p_S : \mathfrak{X} \to \mathbb{R}_+ \times S, \quad p_S(x,h) = (\sqrt{x^2 + ||\pi_S^\perp(h)||^2}, \pi_S(h)).$$

We have a homeomorphism

$$\mathfrak{X} \to \lim_{S \in \mathcal{F}(V)} \mathbb{R}_+ \times S, \quad (x,h) \mapsto (p_S(x,h))_S. \tag{4.5.20}$$

Observe that if $S \in \mathcal{F}(V)$, then the subspace topology on $\mathbb{R}_+ \times S \subset \mathfrak{X}$ is the usual Euclidean topology. Let $v \in S \in \mathcal{F}(V)$ and let

$$\overset{\circ}{D}_S((r,v),\delta) = \{(x,s) : (x-r)^2 + ||s-v||^2 < \delta^2\}$$

be the open ball in $\mathbb{R}_+ \times S$. The subsets

$$U(S,r,v,\delta) = p_S^{-1}(\overset{\circ}{D}_S((r,v),\delta)) \quad (S \in \mathcal{F}(V), (r,v) \in \mathbb{R}_+ \times S, \delta > 0), \tag{4.5.21}$$

are open and form a sub-basis for the topology of $\mathfrak{X}$. Observe that the maps

$$\mathcal{Z}_c(S) \to C_c(\mathfrak{X}), \quad f \mapsto f p_S$$

induce a monomorphism

$$\mathcal{Z}_c(V) \hookrightarrow C_c(\mathfrak{X}). \tag{4.5.22}$$

Its image consists of those f which factor through a projection p_S.

Let $S \in \mathcal{F}(V)$, $X \subset \mathbb{R}_+ \times S$ a locally closed subset. Put

$$\mathcal{Z}_c(X) = C_c(X).$$

If X happens to be open then $\mathcal{Z}_c(X)$ is the subalgebra of $\mathcal{Z}_c(S)$ consisting of those elements f with $\mathrm{supp}(f) \subset X$.

Lemma 4.5.23. *Let $S \in \mathcal{F}(V)$, $X \subset \mathbb{R}_+ \times S$ a locally closed subset, and let $Z \subset X$ be closed in the subspace topology. Then the restriction map $\mathcal{Z}_c(X) \to \mathcal{Z}_c(Z)$ is surjective.*

Proof. This is a straightforward application of Tietze's extension theorem. □

For $X \supset Z$ as in Lemma 4.5.23, we write

$$I(X,Z) = \ker(\mathcal{Z}_c(X) \to \mathcal{Z}_c(Z)).$$

The following trivial observation will be useful in what follows.

Lemma 4.5.24. *Let $S \subset T \in \mathcal{F}(V)$ and $X \subset \mathbb{R}_+ \times S$. Then $p_S^{-1}(X) \cap (\mathbb{R}_+ \times T) = p_{ST}^{-1}(X)$.*

Let $S \in \mathcal{F}(V)$ and let $X \subset \mathbb{R}_+ \times S$ be a locally closed subset. Put $L = p_S^{-1}(X)$; by Lemma 4.5.24, if $T' \supset T \supset S$, then (4.5.18) defines a map $\beta_{T'T} : \mathcal{Z}_c((\mathbb{R}_+ \times T) \cap L) \to \mathcal{Z}_c((\mathbb{R}_+ \times T') \cap L)$. Write

$$\mathcal{Z}_c(L) = \operatorname*{colim}_{T \supset S} \mathcal{Z}_c((\mathbb{R}_+ \times T) \cap L). \tag{4.5.25}$$

If $Z \subset X$ is closed in the subspace topology, and $M = p_S^{-1}(Z)$, we write $I(L, M) = \ker(\mathcal{Z}_c(L) \to \mathcal{Z}_c(M))$. We have

$$I(L, M) = \operatorname*{colim}_{T \supset S} I((\mathbb{R}_+ \times T) \cap L, (\mathbb{R}_+ \times T) \cap M). \tag{4.5.26}$$

If now G acts on V by affine isometries, then the action extends by continuity to an action by affine isometries on $\mathfrak{H}$. Let G act on $\mathfrak{X}$ via $g(x, h) = (x, gh)$. We also have an action of G on $\lim_S(\mathbb{R}_+ \times S)$ via

$$(g(x_S, v_S))_{gS} = (x_S, g(v_S));$$

one checks that the homeomorphism (4.5.20) is equivariant with respect to these actions. Similarly, the map (4.5.22) is a homomorphism in G-$\mathfrak{BC}^*$. We remark that if the action of G on V is metrically proper then so are the actions on $\mathfrak{H}$ and on $\mathbb{R} \times \mathfrak{H}$. In particular by (4.5.13), we have

$$\lim_{g \to \infty} ||g(r, v)|| = \infty \quad (r, v) \in \mathbb{R}_+ \times \mathfrak{H}. \tag{4.5.27}$$

The following lemma is an immediate consequence of (4.5.27).

Lemma 4.5.28. *Let G act on V by affine isometries. Assume that the action is metrically proper. Let $X, Y \subset \mathbb{R}_+ \times \mathfrak{H}$ be bounded subsets and let $\mathcal{G} \subset G$ be a finite subset. Then the set*

$$\tilde{\mathcal{G}}_{X,Y} = \{h \in G : \mathcal{G}X \cap hY \neq \emptyset\}$$

is finite.

Let $\mathfrak{X}$ be as in (4.5.19). In (4.5.21) we have introduced the open subsets $U(S, r, v, \delta) \subset \mathfrak{X}$. We shall also consider the compact subsets

$$W(S, r, v, \delta) = p_S^{-1}(D_S((r, v), \delta)) \quad (S \in \mathcal{F}(V), (r, v) \in \mathbb{R}_+ \times S, \delta > 0). \tag{4.5.29}$$

Consider the stabilizer subgroup of an element $v \in V$:

$$G_v = \{g \in G : gv = v\}.$$

If the action of G on V is metrically proper, then G_v is finite for all $v \in V$.

Lemma 4.5.30. *Let $\mathfrak{X}$ be as in (4.5.19) and let $(r, v) \in \mathbb{R}_+ \times V$. Let G act on V by affine isometries. Assume that the action is metrically proper. Then there exist a precompact open neighborhood $(r, v) \in U \subset \mathfrak{X}$ and an affine subspace $S \in \mathcal{F}(V)$ such that*

i) $U = p_S^{-1}(U \cap (\mathbb{R}_+ \times S))$.

ii)

$$gU \cap U = \begin{cases} U & g \in G_v \\ \emptyset & g \notin G_v \end{cases}$$

Proof. Let $S_1 \in \mathcal{F}(V)$ such that $v \in S_1$. Because G_v is finite, the affine subspace S_1' generated by the orbit $G_v S_1$ is in $\mathcal{F}(V)$. Hence, upon replacing S_1 by S_1' if necessary, we may assume that

$$S_1 = G_v S_1. \tag{4.5.31}$$

Let $\delta > 0$ and let $W = W(S_1, r, v, \delta)$. By definition, an element $(x, h) \in \mathfrak{X}$ is in W if and only if

$$\delta^2 \geq (\sqrt{x^2 + ||\pi^{\perp}_{S_1}(h)||^2} - r)^2 + ||\pi_{S_1}(h) - v||^2. \tag{4.5.32}$$

We may rewrite the right hand side of (4.5.32) as

$$x^2 + ||h - v||^2 + r^2 - 2r\sqrt{x^2 + ||\pi^{\perp}_{S_1}(h)||^2}. \tag{4.5.33}$$

Observe that if $g \in G_v$, then

$$||gh - v|| = ||gh - gv|| = ||h - v||.$$

Moreover for $\ell = \ell_g$ as in (4.4.9), using (4.5.31) in the second identity, we have

$$||\pi^{\perp}_{S_1}(gh)|| = ||\ell(\pi^{\perp}_{g^{-1}S_1}(h))|| = ||\pi^{\perp}_{S_1}(h)||.$$

We have shown that $G_v W = W$. Observe also that the expression (4.5.33) goes to infinity as $||h||$ does. In particular the map $(x, h) \to ||h||$ is bounded on W. Hence by (4.5.13) $W \cap G(r, v)$ is finite. Taking δ sufficiently small, we obtain $W \cap G(r, v) = \{(r, v)\}$. By Lemma 4.5.28, the set $\mathcal{G} = \{g \in G : W \cap gW \neq \emptyset\} \setminus G_v$ is finite. Let $U_1 = U(S_1, r, v, \delta)$; put

$$U = U_1 \setminus (\mathcal{G}W).$$

Let $S = \mathcal{G}S_1$. Then U is precompact and satisfies both (i) and (ii). $\square$

An open set $U \subset \mathfrak{X}$ is called *G-admissible* if it admits a finite open covering

$$U = \bigcup_{i=1}^{n} U_i \tag{4.5.34}$$

such that each U_i is precompact and satisfies the conditions of Lemma 4.5.30 for some $(r_i, v_i) \in U_i$.

Let $U \subset \mathfrak{X}$ be an open subset. Assume that there exists $S \in \mathcal{F}(V)$ such that $U = p_S^{-1}(U \cap (\mathbb{R}_+ \times S))$. Then if $\mathcal{G} \subset G$ is finite and $T \supset \mathcal{G}S$, we have $\mathcal{G}U = p_T^{-1}((\mathcal{G}U) \cap (\mathbb{R}_+ \times T))$. Hence the algebra $\mathcal{Z}_c(\mathcal{G}U)$ is defined by (4.5.25). Put

$$\mathcal{Z}_c(G, U) = \operatorname*{colim}_{\mathcal{G} \subset G} \mathcal{Z}_c(\mathcal{G}U). \tag{4.5.35}$$

Here the colimit runs over the finite subsets $\mathcal{G} \subset G$.

Lemma 4.5.36. *Let G be a discrete group acting on V by affine isometries. Assume that the action is metrically proper. Then*

$$\mathcal{Z}_c(V) = \operatorname*{colim}_{U} \mathcal{Z}_c(G, U),$$

where the colimit runs over the G-admissible open subsets of $\mathfrak{X}$.

Proof. Let $U_\rho = U(0,0,0,\rho) \subset \mathfrak{X}$. We have $U_\rho \cap (\mathbb{R}_+ \times S) = \overset{\circ}{D}_S((0,0),\rho)$ for every $0 \in S \in \mathcal{F}(V)$. Thus

$$\mathcal{Z}_c(V) = \operatorname*{colim}_{0 \in S} \operatorname*{colim}_{\rho} \mathcal{Z}_c(\overset{\circ}{D}_S((0,0),\rho)) = \operatorname*{colim}_{\rho} \mathcal{Z}_c(U_\rho).$$

Because $U_\rho \subset W(0,0,0,\rho)$, which is compact, there is a G-admissible open subset $U_\rho \subset U \subset \mathfrak{X}$ $(\rho > 0)$, by Lemma 4.5.30. On the other hand, since a G-admissible open set is precompact by definition, it is bounded, whence it is contained in some U_ρ. Hence

$$\operatorname*{colim}_{\rho} \mathcal{Z}_c(U_\rho) = \operatorname*{colim}_{U} \mathcal{Z}_c(U) = \operatorname*{colim}_{U} \mathcal{Z}_c(G, U),$$

where the last two colimits are taken over the G-admissible open sets $U \subset \mathfrak{X}$. This completes the proof. □

Let $U \subset \mathfrak{X}$ be a G-admissible open subset and let $\mathscr{U} = \{U_1, \dots, U_n\}$ and $v_1, \dots, v_n$ be as in (4.5.34). We may choose $S \in \mathcal{F}(V)$ such that

$$U_i = p_S^{-1}(U_i \cap (\mathbb{R}_+ \times S)), \quad (i = 1, \dots, n). \tag{4.5.37}$$

Write

$$G_i = G_{v_i}, \quad U_{<i} = \bigcup_{j<i} U_j.$$

Let $\mathcal{G} \subset G$ be a finite subset. With the notations of (4.5.25) and of Lemma 4.5.28, put

$$\begin{gathered} \tilde{\mathcal{G}}^i = \tilde{\mathcal{G}}_{U, U_{<i}}, \\ \mathcal{Z}_c^i(\mathcal{G}, \mathscr{U}) = \mathcal{Z}_c(\mathcal{G}U \setminus \tilde{\mathcal{G}}^i U_{<i}). \end{gathered}$$

Observe that

$$\begin{gathered} i < j \Rightarrow \tilde{\mathcal{G}}^i \subset \tilde{\mathcal{G}}^j, \\ \mathcal{G}U \setminus \tilde{\mathcal{G}}^i U_{<i} = \mathcal{G}U \setminus \tilde{\mathcal{G}}^j U_{<i} \supset \mathcal{G}U \setminus \tilde{\mathcal{G}}^j U_{<j}. \end{gathered} \tag{4.5.38}$$

Moreover, $\mathcal{G}U \setminus \tilde{\mathcal{G}}^j U_{<j}$ is closed in $\mathcal{G}U \setminus \tilde{\mathcal{G}}^i U_{<i}$ $(i < j)$. With the notation of (4.5.26), put

$$J^i(\mathcal{G}, \mathscr{U}) = I(\mathcal{G}U \setminus \tilde{\mathcal{G}}^i U_{<i}, \mathcal{G}U \setminus \tilde{\mathcal{G}}^{i+1} U_{<i+1}). \tag{4.5.39}$$

By Lemma 4.5.23 we have an exact sequence

$$0 \to J^i(\mathcal{G}, \mathscr{U}) \to \mathcal{Z}_c^i(\mathcal{G}, \mathscr{U}) \to \mathcal{Z}_c^{i+1}(\mathcal{G}, \mathscr{U}) \to 0.$$

Note that

$$\mathcal{Z}_c^{n+1}(\mathcal{G}, \mathscr{U}) = 0, \quad J^n(\mathcal{G}, \mathscr{U}) = \mathcal{Z}_c^n(\mathcal{G}, \mathscr{U}). \tag{4.5.40}$$

If $\mathcal{H} \supset \mathcal{G}$ is another finite subset of G, then $\mathcal{G}U \setminus (\tilde{\mathcal{G}}^i U_{<i})$ is open in $\mathcal{H}U \setminus (\tilde{\mathcal{H}}^i U_{<i})$. Hence $\mathcal{Z}_c^i(\mathcal{G}, \mathscr{U}) \subset \mathcal{Z}_c^i(\mathcal{H}, \mathscr{U})$ and thus the algebraic colimit

$$\mathcal{Z}_c^i(G, \mathscr{U}) = \operatorname*{colim}_{\mathcal{G}} \mathcal{Z}_c^i(\mathcal{G}, \mathscr{U}) \tag{4.5.41}$$

is in G-$\mathfrak{BC}^*$. One checks that restrictions induce an equivariant map $\mathcal{Z}_c^i(G, \mathscr{U}) \to \mathcal{Z}_c^{i+1}(G, \mathscr{U})$, and so for

$$J^i(G, \mathscr{U}) = \operatorname*{colim}_{\mathcal{G}} J^i(\mathcal{G}, \mathscr{U}) \tag{4.5.42}$$

we have an exact sequence in G-$\mathfrak{BC}^*$

$$0 \to J^i(G, \mathscr{U}) \to \mathcal{Z}_c^i(G, \mathscr{U}) \to \mathcal{Z}_c^{i+1}(G, \mathscr{U}) \to 0. \tag{4.5.43}$$

Lemma 4.5.44. *Let $J^i(G, \mathscr{U}) \in G$-$\mathfrak{BC}^*$ be as in* (4.5.42) *($1 \leq i \leq n$). Then $J^i(G, \mathscr{U})$ is proper over G/G_i.*

Proof. Let $S \in \mathcal{F}(V)$ be as in (4.5.37), and let $\mathcal{G} \subset G$ be a finite subset. By (4.5.26) and (4.5.39), $J^i(\mathcal{G}, \mathscr{U})$ is the colimit, over $T \supset \tilde{\mathcal{G}}^{i+1}S$, of the ideals $J^i(\mathcal{G}, \mathscr{U}, T) \lhd \mathcal{Z}_c((\mathbb{R}_+ \times T) \cap (\mathcal{G}U \setminus \tilde{\mathcal{G}}^i U_{<i}))$ of those functions f which vanish outside $\tilde{\mathcal{G}}^{i+1}U_i$. Let $\overline{\mathcal{G}^{i+1}}$ be the image of $\tilde{\mathcal{G}}^{i+1}$ in G/G_i. By our hypothesis on U_i, $\tilde{\mathcal{G}}^{i+1}U_i$ is the disjoint union of the open subsets $\bar{g}U_i$ ($\bar{g} \in \overline{\mathcal{G}^{i+1}}$). Let $J^i(\mathcal{G}, \mathscr{U}, T)_{\bar{g}} \subset J^i(\mathcal{G}, \mathscr{U}, T)$ be the subalgebra of those functions f which vanish outside $\bar{g}U_i$. Then

$$J^i(\mathcal{G}, \mathscr{U}, T) = \bigoplus_{\bar{g} \in \overline{\mathcal{G}^{i+1}}} J^i(\mathcal{G}, \mathscr{U}, T)_{\bar{g}}$$
$$\text{and } J^i(\mathcal{G}, \mathscr{U}, T)_{\bar{g}} J^i(\mathcal{G}, \mathscr{U}, T)_{\bar{h}} = 0 \text{ if } \bar{g} \neq \bar{h}.$$

Hence $J^i(\mathcal{G}, \mathscr{U}, T)$ is an algebra over $\mathbb{C}^{(\overline{\mathcal{G}^{i+1}})}$ such that

$$C^{(\overline{\mathcal{G}^{i+1}})} J^i(\mathcal{G}, \mathscr{U}, T) = J^i(\mathcal{G}, \mathscr{U}, T).$$

One checks that this structure is compatible with the maps $J^i(\mathcal{G}, \mathscr{U}, T) \to J^i(\mathcal{G}, \mathscr{U}, T')$, and so we get a $\mathbb{C}^{(\overline{\mathcal{G}^{i+1}})}$-algebra structure on $J^i(\mathcal{G}, \mathscr{U})$ with

$$\mathbb{C}^{(\overline{\mathcal{G}^{i+1}})} J^i(\mathcal{G}, \mathscr{U}) = J^i(\mathcal{G}, \mathscr{U}).$$

Passing to the colimit along the inclusions $\mathcal{G} \subset \mathcal{H}$ one obtains a full compatible $(G, \mathbb{C}^{(G/G_i)})$-algebra structure on $J^i(G, \mathscr{U})$. □

Proof of Theorem 4.5.14. Because $\mathcal{Z}_c(V) \subset \mathcal{A}_c(V)$ is a central G-subalgebra, $\mathcal{A}_c(V)$ carries a canonical compatible $(G, \mathcal{Z}_c(V))$-structure. Moreover, by (4.5.15) we have

$$\mathcal{A}_c(V) = \mathcal{Z}_c(V)\mathcal{A}_c(V). \tag{4.5.45}$$

Condition (4.5.5) also holds because it holds for the action of $\mathcal{Z}_c(S)$ on $\mathcal{A}_c(S)$ ($S \in \mathcal{F}(V)$). Let $U \subset \mathfrak{X}$ be a G-admissible open subset. Put

$$\mathcal{A}_c(G, U) = \mathcal{Z}_c(G, U)\mathcal{A}_c(V).$$

Because we are assuming that $\mathbb{E}$ and $\mathbb{F}$ commute with filtering colimits up to homotopy, it suffices, in view of Lemma 4.5.36, to prove that $\tau(\mathcal{A}_c(G, U))$ is a weak equivalence for every

G-admissible open subset $U \subset \mathfrak{X}$. Let $\mathscr{U} = \{U_1, \dots, U_n\}$ be as in (4.5.34). Define inductively

$$\mathcal{A}_c^1(G,\mathscr{U}) = \mathcal{A}_c(G,U), \quad I^i(G,\mathscr{U}) = J^i(G,\mathscr{U})\mathcal{A}_c^i(G,\mathscr{U}),$$
$$\mathcal{A}_c^{i+1}(G,\mathscr{U}) = \mathcal{A}_c^i(G,\mathscr{U})/I^i(G,\mathscr{U}).$$

By (4.5.40), we have

$$\mathcal{A}_c^{n+1}(G,\mathscr{U}) = 0, \quad I^n(G,\mathscr{U}) = \mathcal{A}_c^n(G,\mathscr{U}).$$

Hence because we are assuming that $\mathbb{E}$ and $\mathbb{F}$ satisfy excision, by (4.5.43) and induction, we can further reduce to proving that $\tau(I^i(G,\mathscr{U}))$ is a weak equivalence ($1 \le i \le n$). This follows from Lemma 4.5.11, Lemma 4.5.44, and the hypothesis that $\tau(P)$ is a weak equivalence whenever P is proper over G/H and H is finite. □

4.6 Main results

Let G be a group and $\mathcal{F}in$ the family of its finite subgroups. An equivariant map $f : X \to Y$ of G-spaces is called a *$\mathcal{F}in$-equivalence* if $f : X^H \to Y^H$ is a weak equivalence for $H \in \mathcal{F}in$.

Theorem 4.6.1. *Let G be a countable discrete group. Let $B \in G\text{-}\mathfrak{BC}^*$, let I be a K-excisive G-ring, let $\underset{\sim}{\otimes}$ be the spatial tensor product, and let $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ be the algebra of compact operators; equip $\mathcal{K}$ with the trivial G-action. Assume that G acts metrically properly by affine isometries on a countably infinite dimensional Euclidean space V. Then the functor $H^G(-, K(I \otimes (B\underset{\sim}{\otimes}\mathcal{K})))$ sends $\mathcal{F}in$-equivalences of G-spaces to weak equivalences of spectra.*

Proof. Let Z be a G-space; consider the functor $\mathbb{E}_Z : G\text{-}\mathfrak{BC}^* \to \mathrm{Spt}$,

$$\mathbb{E}_Z(A) = H^G(Z, K(I \otimes (A \otimes_\mu \mathcal{K} \otimes_\mu B))).$$

We must prove that if $X \to Y$ is a $\mathcal{F}in$-equivalence then $\mathbb{E}_X(\mathbb{C}) \to \mathbb{E}_Y(\mathbb{C})$ is a weak equivalence. By Corollary 4.4.12 it suffices to show that $\mathbb{E}_X(\mathcal{A}_c(V)) \to \mathbb{E}_Y(\mathcal{A}_c(V))$ is a weak equivalence. By Theorem 4.2.2.4, the functor $\mathbb{E}_Z$ is excisive, homotopy invariant and G-stable. Moreover, it commutes with filtering colimits along injective maps up to weak equivalence, since algebraic K-theory commutes with arbitrary filtering algebraic colimits up to weak equivalence. Therefore, by Theorem 4.5.14 we are reduced to proving that if $H \subset G$ is a finite subgroup and $A \in G\text{-}\mathfrak{BC}^*$ is proper over G/H, then

$$\mathbb{E}_X(A) \to \mathbb{E}_Y(A) \tag{4.6.2}$$

is a weak equivalence. By Lemma 4.5.10, $C = A \otimes_\mu \mathcal{K} \otimes_\mu B$ is proper over G/H as a $*$-algebra, and thus $I \otimes C$ is proper over G/H as a ring. This finishes the proof, since we know from [6, Proposition 4.3.1, Lemma 11.1, and Theorem 11.6], that if H is finite and J is a K-excisive G-ring, proper over G/H, then $H^G(-, K(J))$ maps $\mathcal{F}in$-equivalences to weak equivalences. □

Corollary 4.6.3. *(Farrell-Jones' conjecture) Let G, I, B and $\mathcal{K}$ be as in Theorem 4.6.1. Then the assembly map*

$$H^G(\mathcal{E}(G, \mathcal{V}cyc), K(I \otimes (B\underset{\sim}{\otimes}\mathcal{K}))) \to K((I \otimes (B\underset{\sim}{\otimes}\mathcal{K})) \rtimes G)$$

is a weak equivalence.

Proof. The assembly map is induced by $\mathcal{E}(G, \mathcal{V}cyc) \to pt$, which is a $\mathcal{V}cyc$-equivalence, and therefore a $\mathcal{F}in$-equivalence. $\square$

If $\mathfrak{B}$ is a C^*-algebra then by Suslin-Wodzicki's theorem (Karoubi's conjecture) [31, Theorem 10.9] and stability of K^{top}, we have a weak equivalence

$$K(\mathfrak{B}\underset{\sim}{\otimes}\mathcal{K}) \xrightarrow{\sim} K^{\mathrm{top}}(\mathfrak{B}\underset{\sim}{\otimes}\mathcal{K}) \xleftarrow{\sim} K^{\mathrm{top}}(\mathfrak{B}).$$

If G is a group and $\mathfrak{A}$ is a G-C^*-algebra then

$$(\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K}) \rtimes G \subset C^*_{\mathrm{red}}(G, \mathfrak{A}\underset{\sim}{\otimes}\mathcal{K}) \cong C^*_{\mathrm{red}}(G, \mathfrak{A})\underset{\sim}{\otimes}\mathcal{K}.$$

Thus there is a map

$$K((\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K}) \rtimes G) \to K^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A})). \tag{4.6.4}$$

Corollary 4.6.5. *Let G be as in Theorem 4.6.1 and let $\mathfrak{A}$ be a separable G-C^*-algebra. Then* (4.6.4) *is a weak equivalence.*

Proof. We have a homotopy commutative diagram

$$\begin{tikzcd} H^G(\mathcal{E}(G, \mathcal{F}in), K(\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K})) \arrow[r] \arrow[d] & K((\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K}) \rtimes G) \arrow[d] \\ H^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathfrak{A})) \arrow[r] & K^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A})). \end{tikzcd} \tag{4.6.6}$$

By Corollary 4.6.3 the top horizontal arrow in (4.6.6) is a weak equivalence. By [17, Corollary 8.4], the bottom arrow is equivalent to the Baum-Connes assembly map, which is an equivalence for Haagerup groups, by [20, Theorem 1.1]. It follows from the Suslin-Wodzicki theorem [31, Theorem 10.9] that the map (4.6.4) is an equivalence for finite G. Since $\mathcal{E}(G, \mathcal{F}in)$ has finite stabilizers, the latter fact implies that the vertical map on the left is a weak equivalence. This concludes the proof. $\square$

4.7 Resumen

En este capítulo estudiamos la conjetura de isomorfismo para K-teoría con anillos de coeficientes de la forma

$$R = I \otimes (\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K}), \tag{4.7.1}$$

donde I es un G-anillo K-escisivo, $\mathfrak{A}$ es una G-C^*-álgebra compleja (o más generalmente, una C^*-álgebra bornolocal como se define en la Sección 4.1), $\underset{\sim}{\otimes}$ es el producto tensorial espacial (ver Apéndice A.2.1), y $\mathcal{K}$ es el ideal de operadores compactos en un espacio de Hilbert sobre $\mathbb{C}$ de dimensión infinita, separable, con acción trivial de G. Consideramos la conjetura de Farrell-Jones para grupos discretos con la *propiedad de aproximación de Haagerup*. Éstos son los grupos discretos numerables que admiten una acción afín, isométrica y *métricamente propia* en un espacio de pre-Hilbert real V de dimensión numerable (o equivalentemente, en un espacio de Hilbert). Decimos que la acción es métricamente propia si para todo $v \in V$,

$$\lim_{g\to\infty} ||gv|| = \infty.$$

Gromov llamó a estos grupos *a-T-menables* ([15]). Nuestro resultado principal es el siguiente (ver Teorema 4.6.1).

Teorema 4.7.2. *Sea G un grupo discreto numerable. Sean $\mathfrak{A}$ una G-C^*-álgebra, I un G-anillo, y sea $\mathcal{K} = \mathcal{K}(\ell^2(\mathbb{N}))$ el álgebra de operadores compactos equipada con la acción trivial de G. Supongamos que I es K-escisivo y que G tiene la propiedad de aproximación de Haagerup. Entonces el funtor $H^G(-, K(I \otimes (\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K})))$ manda $\mathcal{F}in$-equivalencias de G-espacios en equivalencias débiles de espectros.*

Dado que $\mathcal{V}cyc \supset \mathcal{F}in$, toda $\mathcal{V}cyc$-equivalencia es también una $\mathcal{F}in$-equivalencia. Por ser $\mathcal{E}(G, \mathcal{V}cyc) \to pt$ una $\mathcal{V}cyc$-equivalencia por definición, el teorema tiene el siguiente corolario (ver Corolario 4.6.3).

Corolario 4.7.3. *Sean G, I y $\mathfrak{A}$ como en el Teorema 4.7.2. Entonces G satisface la conjetura de Farrell-Jones para K-teoría con coeficientes en $I \otimes (\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K})$.*

Higson y Kasparov probaron en [20] que los grupos que tienen la propiedad de aproximación de Haagerup satisfacen la *conjetura de Baum-Connes* con coeficientes en cualquier G-C^*-álgebra separable. Recordemos que esta conjetura es el análogo de la conjetura de Farrell-Jones para la K-teoría topológica de productos cruzados reducidos de C^*-álgebras (para la definición de productos cruzados ver Apéndice C). Esta conjetura afirma que el morfismo de ensamble

$$H^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathfrak{A})) \to K^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A}))$$

es una equivalencia débil.

La aplicación natural

$$\mathfrak{A} \rtimes H \to C^*_{\mathrm{red}}(H, \mathfrak{A}) \tag{4.7.4}$$

resulta un isomorfismo si H es finito. Tenemos además el siguiente diagrama, que conmuta salvo homotopía

$$\begin{tikzcd}
H^G(\mathcal{E}(G, \mathcal{F}in), K(\mathfrak{A})) \arrow[r] \arrow[d] & K(\mathfrak{A} \rtimes G) \arrow[d] \\
H^G(\mathcal{E}(G, \mathcal{F}in), K^{\mathrm{top}}(\mathfrak{A})) \arrow[r] & K^{\mathrm{top}}(C^*_{\mathrm{red}}(G, \mathfrak{A})).
\end{tikzcd} \tag{4.7.5}$$

Del teorema de Suslin y Wodzicki (conjetura de Karoubi) ([31, Theorem 10.9]) y de los hechos de que (4.7.4) es un isomorfismo para H finito, y que G actúa en $\mathcal{E}(G, \mathcal{F}in)$ con estabilizadores finitos, se sigue que la aplicación vertical de la izquierda en (4.7.5) es una equivalencia débil cuando $\mathfrak{A}$ es de la forma $\mathfrak{A} = \mathfrak{B}\underset{\sim}{\otimes}\mathcal{K}$. Usando esto, la estabilidad de K^{top} al tensorizar con $\mathcal{K}$, y el resultado de Higson y Kasparov, obtenemos el siguiente corolario del Teorema 4.7.2 (ver Corolario 4.6.5).

Corolario 4.7.6. *Sean G y $\mathfrak{A}$ como en el Teorema 4.7.2. Supongamos que $\mathfrak{A}$ es separable. Entonces se tiene un isomorfismo:*

$$K_*((\mathfrak{A}\underset{\sim}{\otimes}\mathcal{K}) \rtimes G) \cong K_*^{\mathrm{top}}(C^*_{\mathrm{red}}(G,\mathfrak{A})).$$

Higson y Kasparov mostraron en [20, Teorema 7.1] que si G es un grupo localmente compacto con la propiedad de Haagerup, $\mathfrak{A}$ es una G-C^*-álgebra separable, y $C^*(G,\mathfrak{A})$ es el producto cruzado pleno, entonces la aplicación

$$K_*^{\mathrm{top}}(C^*(G,\mathfrak{A})) \to K_*^{\mathrm{top}}(C^*_{\mathrm{red}}(G,\mathfrak{A}))$$

es un isomorfismo. Por lo tanto en el Corolario 4.7.6 podemos sustituir el producto cruzado reducido de C^*-álgebras por su versión plena.

El resto del capítulo está organizado de la siguiente manera. En la Sección 4.1 presentamos las C^*-álgebras bornolocales. Éstas son $*$-álgebras normadas sobre $\mathbb{C}$ tales que $\|a^*a\| = \|a\|^2$, no necesariamente completas, que son uniones filtrantes de C^*-subálgebras. Por ejemplo, si X es un espacio de Hausdorff localmente compacto y $\mathfrak{A}$ es una C^*-álgebra, entonces el álgebra $C_c(X,\mathfrak{A})$ de funciones continuas de soporte compacto de X en $\mathfrak{A}$ es una C^*-álgebra bornolocal. Llamamos $\mathfrak{BC}^*$ a la categoría de C^*-álgebras bornolocales y G-$\mathfrak{BC}^*$ a la categoría equivariante correspondiente. En la Sección 4.2 probamos el Teorema 4.2.2.4, que afirma que si G es un grupo discreto, X un G-espacio, I un G-anillo K-escisivo y $B \in G$-$\mathfrak{BC}^*$, entonces el funtor

$$\mathbb{E}_X : G\text{-}\mathfrak{BC}^* \to \mathrm{Spt}, \quad A \mapsto H^G(X, K(I \otimes A \otimes_\mu \mathcal{K} \otimes_\mu B)) \tag{4.7.7}$$

es escisivo, invariante homotópico, y G-estable. Aquí $\otimes_\mu$ es el producto tensorial maximal de C^*-álgebras bornolocales; está definido en (4.2.2.1). La Sección 4.3 está dedicada a los morfismos asintóticos de G-C^*-álgebras bornolocales. En esta sección técnica, discutimos cómo extender el funtor (4.7.7) a un funtor $\bar{\mathbb{E}}_X$ que se pueda aplicar a ciertos morfismos asintóticos equivariantes; los resultados principales de esta sección son la Proposición 4.3.3.7 y el Lema 4.3.3.13. En la Sección 4.4 recordamos la contrucción de Higson-Kasparov de un elemento dual de Dirac en E-teoría equivariante ([20]). Para un grupo G que actúa por isometrías afines en un espacio euclídeo de dimensión infinita numerable V, construyen una G-C^*-álgebra $\mathcal{A}_0(V)$ que es un C^*-colímite sobre todos los subespacios de dimensión finita $S \subset V$, de álgebras de funciones continuas $\mathbb{R} \times S \to \mathrm{Cliff}(\mathbb{R} \oplus S)$, que se anulan en el infinito, y que toman valores en el álgebra de Clifford complexificada $\mathrm{Cliff}(\mathbb{R} \oplus S)$. Definen un morfismo asintótico equivariante

$$\hat{\beta}_0 : C_0(\mathbb{R}) -- > \mathcal{A}_0(V), \tag{4.7.8}$$

y muestran que su clase en $E_G(C_0(\mathbb{R}), \mathcal{A}_0(V))$, a la que llaman *elemento dual de Dirac*, es inversible. Nosotros definimos una G-C^*-álgebra bornolocal $\mathcal{A}_c(V)$ que es un colímite algebraico, sobre los subespacios de dimensión finita $S \subset V$, de álgebras de funciones continuas de soporte compacto $\mathbb{R} \times S \to \mathrm{Cliff}(\mathbb{R} \oplus S)$. El morfismo (4.7.8) se restringe a un morfismo asintótico equivariante

$$\hat{\beta}_c : C_c(\mathbb{R}) -- > \mathcal{A}_c(V).$$

Usando el resultado de Higson y Kasparov, junto con el Lema 4.3.3.13, mostramos en la Proposición 4.4.10 que para la extensión $\bar{\mathbb{E}}_X$ de la Sección 4.3, $\bar{\mathbb{E}}_X(\hat{\beta}_c)$ tiene una inversa homotópica a izquierda. Luego en el Corolario 4.4.12 damos la siguiente aplicación. Sean $f : X \to$

Y una aplicación equivariante y

$$\mathbb{E}_X \to \mathbb{E}_Y \tag{4.7.9}$$

la transformación natural inducida por f. Luego

$$\mathbb{E}_X(\mathbb{C}) \to \mathbb{E}_Y(\mathbb{C})$$

es una equivalencia débil cuando $\mathbb{E}_X(\mathcal{A}_c(V)) \to \mathbb{E}_Y(\mathcal{A}_c(V))$ lo es. En la Sección 4.5 recordamos la noción de G-anillos propios sobre un espacio homogéneo discreto G/H, introducida en [6], que es análoga a la noción de C^*-álgebras propias ([16]). En [16, Teorema 13.1] se muestra que el morfismo de ensamble de Baum-Connes para E-teoría y productos cruzados de C^*-álgebras con coeficientes en G-C^*-álgebras propias es un isomorfismo. El resultado análogo para K-teoría algebraica, productos cruzados algebraicos, $\mathbb{Q}$-álgebras K-escisivas y el morfismo de ensamble de Farell-Jones fue probado en [6, Teorema 13.2.1]. Higson y Kasparov mostraron en [20] que si la acción afín e isométrica de G en V es métricamente propia, luego $\mathcal{A}_0(V)$ es una G-C^*-álgebra propia. En el Teorema 4.5.14 probamos que si

$$\tau : \mathbb{F} \to \mathbb{G}$$

es una transformación natural entre funtores de G-$\mathfrak{BC}^*$ en Spt, entonces la aplicación $\tau(\mathcal{A}_c(V))$ es una equivalencia débil siempre que se satisfagan las siguientes condiciones:

- La acción de G en V es métricamente propia.
- Los funtores $\mathbb{F}$ y $\mathbb{G}$ satisfacen escisión y conmutan salvo equivalencia débil con colímites filtrantes sobre morfismos inyectivos.
- Si $H \subset G$ es un subgrupo finito y P es propia sobre G/H, luego $\tau(P)$ es una equivalencia.

En la Sección 4.6 usamos los resultados anteriores para probar el Teorema 4.7.2 (para C^*-álgebras bornolocales) y los Corolarios 4.7.3 y 4.7.6; éstos son el Teorema 4.6.1 y los Corolarios 4.6.3 y 4.6.5, respectivamente.

Estos resultados fueron publicados en [9].

APPENDIX

A. TENSOR PRODUCTS

A.1 Projective tensor product

Let A and B be normed $\mathbb{C}$-vector spaces. The *projective tensor product* $A \otimes_\pi B$ is the completion of $A \otimes_{\mathbb{C}} B$ with respect to the norm

$$\pi(t) = \inf \left\{ \sum_{i=1}^{n} \|a_i\| \, \|b_i\| \; / \; t = \sum_{i=1}^{n} a_i \otimes b_i \right\}, \quad t \in A \otimes_{\mathbb{C}} B.$$

When A and B are algebras π is submultiplicative and $A \otimes_\pi B$ is a Banach algebra.

A.2 Tensor products of C^*-algebras

The algebraic tensor product $\mathfrak{A} \otimes_{\mathbb{C}} \mathcal{B}$ of two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ has a canonical structure as a $*$-algebra. To make it a C^*-algebra we have to take completions with respect to suitable cross-norms $\|\cdot\|$ satisfying $\|a \otimes b\| = \|a\| \, \|b\|$. Among the possible choices of such norms there is a minimal cross-norm and a maximal cross-norm giving rise to the *spatial tensor product* $\mathfrak{A} \underset{\sim}{\otimes} \mathfrak{B}$ and the *maximal tensor product* $\mathfrak{A} \otimes_\mu \mathfrak{B}$ respectively.

A.2.1 Spatial tensor product

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces. Write $\mathcal{H}_1 \otimes \mathcal{H}_2$ for the completion of $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$ with respect to the unique inner product satisfying

$$< u_1 \otimes v_1, u_2 \otimes v_2 > = < u_1, u_2 > < v_1, v_2 >, \;\; u_i \in \mathcal{H}_1, \; v_i \in \mathcal{H}_2.$$

For each pair of representations ρ_1 and ρ_2 of C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, there is a unique algebraic representation

$$\rho_1 \otimes \rho_2 : \mathfrak{A}_1 \otimes_{\mathbb{C}} \mathfrak{A}_2 \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

of the algebraic tensor product $\mathfrak{A}_1 \otimes_{\mathbb{C}} \mathfrak{A}_2$ as operators on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying

$$(\rho_1 \otimes \rho_2)(a_1 \otimes a_2) = \rho_1(a_1) \otimes \rho_2(a_2) \quad \in \mathcal{B}(\mathcal{H}_1) \otimes_{\mathbb{C}} \mathcal{B}(\mathcal{H}_2) \subset \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2).$$

If both ρ_1 and ρ_2 are injective, then $\rho_1 \otimes \rho_2$ is injective.

The *spatial tensor product* $\mathfrak{A}_1 \underset{\sim}{\otimes} \mathfrak{A}_2$ is the completion of $\mathfrak{A}_1 \otimes_{\mathbb{C}} \mathfrak{A}_2$ with respect to the *spatial C^*-norm* $t \mapsto \|(\rho_1 \otimes \rho_2)(t)\|$, where ρ_i is some faithful representation of $\mathfrak{A}_i$, $i = 1, 2$. This construction is independent of the choice of the representations.

Lemma A.2.1.1. *([32, T.5.19]) If $\phi_i : \mathfrak{A}_i \to \mathfrak{B}_i$ are morphisms of C^*-algebras, then $\phi_1 \otimes \phi_2$ extends by continuity to a morphism $\phi_1 \underset{\sim}{\otimes} \phi_2 : \mathfrak{A}_1 \underset{\sim}{\otimes} \mathfrak{A}_2 \to \mathfrak{B}_1 \underset{\sim}{\otimes} \mathfrak{B}_2$.*

If ϕ_1 and ϕ_2 are both injective, so is $\phi_1 \underset{\sim}{\otimes} \phi_2$.

A.2.2 Maximal tensor product

Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras. Write $\mathcal{R}$ for the set of commuting representations ρ of $\mathfrak{A}$ and τ of $\mathfrak{B}$ on the same Hilbert space (i.e. $\rho(a)\tau(b) = \tau(b)\rho(a)$, for all a in $\mathfrak{A}$ and b in $\mathfrak{B}$). The *maximal tensor product* $\mathfrak{A} \otimes_\mu \mathfrak{B}$ is the completion of $\mathfrak{A} \otimes_{\mathbb{C}} \mathfrak{B}$ with respect to the norm

$$\mu(t) = \sup_{\mathcal{R}} \left\{ \left\| \sum_{i=1}^{n} \rho(a_i)\tau(b_i) \right\| \,/\, t = \sum_{i=1}^{n} a_i \otimes b_i \right\}.$$

Actually the supremum is attained at a particular pair of faithful commuting representations $(\rho, \tau) \in \mathcal{R}$.

A C^*-algebra $\mathfrak{A}$ is called *nuclear* when, for each C^*-algebra $\mathfrak{B}$, there is only one C^*-norm on $\mathfrak{A} \otimes_{\mathbb{C}} \mathfrak{B}$.

B. GRADED C^*-ALGEBRAS

Let $\mathfrak{A}$ be a C^*-algebra. A $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathfrak{A}$ is a decomposition of a $\mathfrak{A}$ into a direct sum of two self-adjoint closed linear subspaces $\mathfrak{A}_0$ and $\mathfrak{A}_1$, such that if $x \in \mathfrak{A}_m$ and $y \in \mathfrak{A}_n$, then $xy \in \mathfrak{A}_{m+n}$ (addition mod 2). An element of $\mathfrak{A}_n$ is said to be *homogeneous of degree* n. The degree of a homogeneous element a is denoted $\deg(a)$. A C^*-subalgebra $\mathfrak{B}$ of a graded C^*-algebra $\mathfrak{A}$ is a *graded C^*-subalgebra* if $\mathfrak{B} = (\mathfrak{B} \cap \mathfrak{A}_0) + (\mathfrak{B} \cap \mathfrak{A}_1)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebras. Define a new product and involution on $\mathfrak{A} \otimes_{\mathbb{C}} \mathfrak{B}$ by

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\deg(b_1).\deg(a_2)}(a_1 a_2 \otimes b_1 b_2),$$
$$(a \hat{\otimes} b)^* = (-1)^{\deg(a).\deg(b)}(a^* \otimes b^*),$$

for homogeneous elementary tensors. The algebraic tensor product with this multiplication and involution is a $*$-algebra, denoted $\mathfrak{A} \hat{\otimes} \mathfrak{B}$.

The *spatial* C^*-algebra completion of $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ is obtained by representing $\mathfrak{A}$ and $\mathfrak{B}$ faithfully on $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert spaces $\mathcal{H}_{\mathfrak{A}}$ and $\mathcal{H}_{\mathfrak{B}}$; then representing $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ via the formula

$$\rho_{\mathfrak{A},\mathfrak{B}}(a \hat{\otimes} b)(v \otimes w) = (-1)^{\deg(b).\deg(v)} \rho_{\mathfrak{A}}(a) v \otimes \rho_{\mathfrak{B}}(b) v;$$

and taking the completion in the operator norm topology.

C. REDUCED AND FULL CROSSED PRODUCTS

Let G be a discrete group. An *action* of G on a C^*-algebra $\mathfrak{A}$ is a group homomorphism from G into the group $Aut(\mathfrak{A})$ of $*$-automorphisms of $\mathfrak{A}$. We denote the action by a dot: $g \mapsto (a \mapsto g \cdot a)$, $\forall g \in G$, $a \in \mathfrak{A}$. A C^*-algebra equipped with an action of G is called a G-C^*-algebra.

Let $\mathfrak{A}$ be a G-C^*-algebra. Write $\mathfrak{A} \rtimes G$ for the algebraic crossed product equipped with the following involution:

$$(a \rtimes g)^* = g^{-1} \cdot a^* \rtimes g^{-1}.$$

A *covariant representation* of a G-C^*-algebra $\mathfrak{A}$ is a triple $(\pi, u, \mathcal{H})$ consisting of a Hilbert space $\mathcal{H}$, a unitary representation $u : G \to \mathcal{B}(\mathcal{H})$ of G and a representation $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ of $\mathfrak{A}$ satisfying the *covariance condition*:

$$u(g)\pi(a)u(g)^* = \pi(g.a), \quad \forall g \in G, \ a \in \mathfrak{A}.$$

It is called nondegenerate if π is nondegenerate. The *integrated form* of $(\pi, u, \mathcal{H})$ is the representation of $\mathfrak{A} \rtimes G$ on $\mathcal{H}$ given by $(\pi \times u)(a \rtimes g) = \pi(a)u(g)$, $\forall a \in \mathfrak{A}$, $g \in G$.

Let $\mathcal{R}$ be the family of all covariant representations of $\mathfrak{A}$. The *full crossed product* $C^*(\mathfrak{A}, G)$ is the completion of $\mathfrak{A} \rtimes G$ with respect to the C^*-norm:

$$\|f\| = \sup_{\mathcal{R}} \left\{ \|(\pi \times u)(f)\| \right\}, \quad f \in \mathfrak{A} \rtimes G.$$

Let $\pi : \mathfrak{A} \to \mathcal{B}(\mathcal{H})$ be a faithful representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$. Consider the left regular representation of G, $\lambda : G \to \mathcal{B}(\ell^2(G, \mathcal{H}))$ given by $\lambda(g)\xi(h) = \xi(g^{-1}h)$, for $g, h \in G$, $\xi \in \ell^2(G, \mathcal{H})$. Write $\tilde{\pi} : \mathfrak{A} \to \mathcal{B}(\ell^2(G, \mathcal{H}))$ for the representation induced by π given by $\tilde{\pi}(a)\xi(g) = \pi(g^{-1} \cdot a)(\xi(g))$. The integrated form of $(\tilde{\pi}, \lambda, \ell^2(G, \mathcal{H}))$ is called the *regular representation* of $\mathfrak{A} \rtimes G$. The *reduced crossed product* $C^*_{red}(\mathfrak{A}, G)$ is the completion of $\mathfrak{A} \rtimes G$ with respect to the C^*-norm:

$$\|f\| = \|(\tilde{\pi} \times \lambda)(f)\|, \quad f \in \mathfrak{A} \rtimes G.$$

This norm does not depend on the choice of π.

D. MULTIPLIERS

The *multiplier algebra* $M(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$ is the largest C^*-algebra which contains $\mathfrak{A}$ as an essential ideal (i.e. $b\mathfrak{A} = \{0\}$ implies $b = 0\ \forall b \in M(\mathfrak{A})$). If $\mathfrak{A}$ is represented faithfully and non-degenerately on a Hilbert space $\mathcal{H}$ (i.e. $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ with $\mathfrak{A}\mathcal{H} = \mathcal{H}$), then $M(\mathfrak{A})$ can be realized as the idealizer

$$M(\mathfrak{A}) = \{T \in \mathcal{B}(\mathcal{H})\ /\ T\mathfrak{A} \cup \mathfrak{A}T \subseteq \mathfrak{A}\}$$

of $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H})$. $M(\mathfrak{A})$ is always unital and $M(\mathfrak{A}) = \mathfrak{A}$ if $\mathfrak{A}$ is unital.

Let $\mathfrak{A}$ be a $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebra and let $A \subset \mathfrak{A}$ be a dense $\mathbb{Z}/2\mathbb{Z}$-graded right submodule. An $\mathfrak{A}$-linear map $D : A \to \mathfrak{A}$ is a *degree one, unbounded self-adjoint multiplier* of $\mathfrak{A}$ iff

- $(Dx)^*y = x^*(Dy)$, $\forall x, y \in A$;
- the operators $D \pm iI$ are isomorphisms from A onto $\mathfrak{A}$; and
- $\deg(Dx) \equiv \deg(x) + 1$, mod 2 $\forall x \in \mathfrak{A}$.

If A is a dense, $\mathbb{Z}/2\mathbb{Z}$-graded, right $\mathfrak{A}$-submodule of $\mathfrak{A}$ and if $D : A \to \mathfrak{A}$ satisfies

- $(Dx)^*y = x^*(Dy)$, $\forall x, y \in A$;
- $\deg(Dx) \equiv \deg(x) + 1$, mod 2 $\forall x \in \mathfrak{A}$; and
- the operators $D \pm iI$ have dense range;

then the closure of D (whose graph is the norm closure in $\mathfrak{A} \oplus \mathfrak{A}$ of the graph of D) is a self-adjoint multiplier. We shall call D an *essentially self-adjoint* multiplier of $\mathfrak{A}$. We will use the same symbol for both D and its closure.

If D is an unbounded self-adjoint multiplier of $\mathfrak{A}$ then the resolvent operators $(D \pm iI)^{-1}$ (viewed as maps of $\mathfrak{A}$ into itself) are bounded multipliers of $\mathfrak{A}$, hence there is a unique *functional calculus* homomorphism

$$C_b(\mathbb{R}) \to M(\mathfrak{A})$$

$$f \to f(D),$$

mapping the resolvent functions $(x \pm i)^{-1}$ to $(D \pm iI)^{-1}$. It has the property that if $g(x) = xf(x)$ then $g(D) = Df(D)$. The functional calculus homomorphism is grading preserving, if we grade $C_b(\mathbb{R})$ by even and odd functions.

E. RESUMEN

En el Apéndice A se encuentran las definiciones de productos tensoriales de C^*-álgebras. Dadas dos C^*-álgebras $\mathfrak{A}$ y $\mathfrak{B}$, el producto tensorial algebraico $\mathfrak{A} \otimes_{\mathbb{C}} \mathfrak{B}$ tiene una estructura canónica de $*$-álgebra. Para dotarlo de una estructura de C^*-álgebra hay que completarlo respecto de alguna norma que satisfaga $\|a \otimes b\| = \|a\| \, \|b\|$. Entre tales normas se destacan una norma minimal, que da lugar al *producto tensorial espacial* $\mathfrak{A} \underset{\sim}{\otimes} \mathfrak{B}$, y una maximal, que da lugar al *producto tensorial maximal* $\mathfrak{A} \otimes_\mu \mathfrak{B}$.

En el Apéndice B se definen las C^*-álgebras graduadas y el producto tensorial espacial graduado.

El Apéndice C está dedicado a los productos cruzados de C^*-álgebras. Dado un grupo discreto G y una G-C^*-álgebra $\mathfrak{A}$, se definen las *representaciones covariantes* de $\mathfrak{A}$ y sus *formas integrales*, que resultan representaciones del producto cruzado algebraico $\mathfrak{A} \rtimes G$. Se consideran dos C^*-normas inducidas por las formas integrales y se completa $\mathfrak{A} \rtimes G$ respecto de tales normas, obteniendo el *producto cruzado pleno* $C^*(\mathfrak{A}, G)$ y el *producto cruzado reducido* $C^*_{red}(\mathfrak{A}, G)$.

Finalmente, en el Apéndice D se definen el *álgebra de multiplicadores* de una C^*-álgebra y los multiplicadores no acotados autoadjuntos. Se introduce también el *cálculo funcional* asociado a tales multiplicadores.

BIBLIOGRAPHY

[1] Arthur Bartels, Tom Farrell, Lowell Jones, and Holger Reich, *On the isomorphism conjecture in algebraic K-theory*, Topology **43** (2004), no. 1, 157–213.

[2] Arthur Bartels and Wolfgang Lück, *Isomorphism conjecture for homotopy K-theory and groups acting on trees*, J. Pure Appl. Algebra **205** (2006), no. 3, 660–696.

[3] Arthur Bartels, Wolfgang Lück, and Holger Reich, *The K-theoretic Farrell-Jones Conjecture for hyperbolic groups*, Invent. Math. **172** (2008), 29-70.

[4] Guillermo Cortiñas, *The obstruction to excision in K-theory and in cyclic homology*, Invent. Math. **164** (2006), no. 1, 143–173.

[5] ______, *Algebraic v. topological K-theory: a friendly match*, Topics in algebraic and topological K-theory, Lecture Notes in Math., vol. 2008, Springer, Berlin, 2011, pp. 103–165.

[6] Guillermo Cortiñas and Eugenia Ellis, *Isomorphism conjectures with proper coefficients*, J. Pure Appl. Algebra **218** (2014), no. 7, 1224–1263.

[7] Guillermo Cortiñas and Gisela Tartaglia, *Operator ideals and assembly maps in K-theory*, Proc. Amer. Math. Soc. **142** (2014), 1089–1099.

[8] ______, *Trace class operators, regulators, and assembly maps in K-theory*, Doc. Math. **19** (2014), 439–455.

[9] ______, *Compact operators and algebraic K-theory for groups which act properly and isometrically on Hilbert space*, J. reine angew. Math., to appear, available at `arXiv:1311.6285v5`.

[10] Guillermo Cortiñas and Andreas Thom, *Comparison between algebraic and topological K-theory of locally convex algebras*, Adv. Math. **218** (2008), no. 1, 266–307.

[11] Joachim Cuntz, *Cyclic theory and the bivariant Chern-Connes character*, Noncommutative geometry, 2004, pp. 73–135.

[12] Joachim Cuntz, Ralf Meyer, and Jonathan M. Rosenberg, *Topological and bivariant K-theory*, Oberwolfach Seminars, vol. 36, Birkhäuser Verlag, Basel, 2007.

[13] Joachim Cuntz and Daniel Quillen, *Excision in bivariant periodic cyclic cohomology*, Invent. Math. **127** (1997), no. 1, 67–98.

[14] James F. Davis and Wolfgang Lück, *Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory*, K-Theory **15** (1998), no. 3, 201–252.

[15] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295.

[16] Erik Guentner, Nigel Higson, and Jody Trout, *Equivariant E-theory for C^*-algebras*, Mem. Amer. Math. Soc. **148** (2000), no. 703, viii+86.

[17] Ian Hambleton and Erik K. Pedersen, *Identifying assembly maps in K- and L-theory*, Math. Ann. **328** (2004), no. 1-2, 27–57.

[18] Nigel Higson, *Algebraic K-theory of stable C^*-algebras*, Adv. in Math. **67** (1988), no. 1, 140.

[19] Nigel Higson and Gennadi Kasparov, *Operator K-theory for groups which act properly and isometrically on Hilbert space*, Electron. Res. Announc. Amer. Math. Soc. **3** (1997), 131–142 (electronic).

[20] ______, *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. **144** (2001), no. 1, 23–74.

[21] Nigel Higson, Gennadi Kasparov, and Jody Trout, *A Bott periodicity theorem for infinite dimensional Euclidean space*, Adv. in Math. **135** (1998), no. 1, 40.

[22] Max Karoubi, *Homologie cyclique et K-théorie*, Astérisque **149** (1987), 147 (French, with English summary).

[23] ———, *Sur la K-théorie multiplicative*, Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), Fields Inst. Commun., vol. 17, Amer. Math. Soc., Providence, RI, 1997, pp. 59–77 (French, with English summary).

[24] Wolfgang Lück, *Chern characters for proper equivariant homology theories and applications to K- and L-theory*, J. Reine Angew. Math. **543** (2002), 193–234.

[25] ———, *The relation between the Baum-Connes conjecture and the trace conjecture*, Invent. Math. **149** (2002), no. 1, 123–152.

[26] Wolfgang Lück and Holger Reich, *The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 703–842.

[27] ———, *Detecting K-theory by cyclic homology*, Proc. London Math. Soc. (3) **93** (2006), no. 3, 593–634.

[28] Randy McCarthy, *The cyclic homology of an exact category*, J. Pure Appl. Algebra **93** (1994), no. 3, 251–296.

[29] John Roe, *Coarse cohomology and index theory on complete Riemannian manifolds*, Mem. Amer. Math. Soc. **104** (1993), no. 497, x+90.

[30] Jonathan Rosenberg, *Comparison between algebraic and topological K-theory for Banach algebras and C^*-algebras*, Handbook of K-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 843–874.

[31] Andrei A. Suslin and Mariusz Wodzicki, *Excision in algebraic K-theory*, Ann. of Math. (2) **136** (1992), no. 1, 51–122.

[32] N. E. Wegge-Olsen, *K-theory and C^*-algebras*, Oxford Science Publications, New York, 1993.

[33] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[34] ———, *Homotopy algebraic K-theory*, Algebraic K-theory and algebraic number theory (Honolulu, HI, 1987), Contemp. Math., vol. 83, Amer. Math. Soc., Providence, RI, 1989, pp. 461–488.

[35] Mariusz Wodzicki, *Algebraic K-theory and functional analysis*, First European Congress of Mathematics, Vol. II (Paris, 1992), 1994, pp. 485–496.

[36] Guoliang Yu, *The algebraic K-theory Novikov conjecture for group algebras over the ring of Schatten class operators*, available at `arXiv:1106.3796`.